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Chapter 5

FROM CELESTIAL MECHANICS TO SPACE FLIGHT MECHANICS - HISTORICAL NOTES ON THE DEVELOPMENT OF ASTRODYNAMICS*

Werner Schulz†

Space flight mechanics has developed from three roots by making use of the methods and results of celestial mechanics, of ballistics, as well as of flight mechanics, navigation and control of aircraft and missiles. Celestial mechanics is a necessity of space flight mechanics because the motions of an artificial satellite around a planet or of a space probe during its free flight outside the atmosphere are subject to the same laws as the motion of a planet around the Sun. Ballistics is of importance because the history of rocket technology springs from the development of artillery. Thirdly, flight mechanics of aircraft is of value to space flight mechanics by offering a suitable system of classification of flight performance and flight characteristics, on the one hand, and methods and results, e.g., on stability behavior, which can be applied in space flight mechanics, on the other hand.

The aim of this paper is to demonstrate by some examples the application of findings in celestial mechanics to problems in space flight mechanics. First of all, classical celestial mechanics is briefly outlined starting with Kepler and Newton, continuing with Euler, Lagrange, Laplace, Gauss, Hamilton, Jacobi and Le Verrier and finally touching on Poincaré, Levi-Civita and Birkhoff.

The second part of the paper will give four examples concerning a) optimum transfer trajectories, b) Laplace's activity sphere of a planet, c) disturbed Kepler orbits, and d) regularization and linearization of the equations of motion - to demonstrate the usefulness of classical celestial mechanics, partly in a direct way and partly in a further developed form for space flight mechanics.

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† Deutsche Gesellschaft für Luft- und Raumfahrt (DGLR), Federal Republic of Germany. Dr. Schulz died in 1984.

ON THE DEVELOPMENT OF CLASSICAL CELESTIAL MECHANICS

Kepler and Newton

For those concerned with the history of space flight mechanics, it is necessary to go back to the beginning of the 17th Century, to Kepler and Newton, who revealed the nature of the planetary system and laid the foundations for celestial mechanics by the laws they discovered.

Johannes Kepler (1571-1630) was the first to accept the ideas of Nicolaus Copernicus (1473-1543), to believe in the Copernican heliocentric system, and to let himself be guided by Copernicus's views in his astronomical investigations. When calculating the planetary tables of Tycho Brahe (1546-1601) and particularly when studying anew the motion of the planet Mars he empirically discovered his first two laws: that the planets describe ellipses around the Sun as a focus and the radius vector Sun-planet sweeps over equal areas in equal times.[26] Further astronomical investigations led Kepler in 1618 to discover the third law, that the squares of the periods of revolution of any planets are proportional to the cubes of the ellipses' semimajor axes.[27]

In 1621 Kepler considered as origin for the motion of the planets a force forthcoming from the Sun. Incidentally, the definitions "eccentricity of an ellipse" as well as "perihelion" and "aphelion" for the points of a planetary orbit nearest, respectively farthest, from the Sun also go back to Kepler.*

The achievement of Isaac Newton (1643-1727) was the discovery of universal gravitation. Certain perceptions of gravitation existed already before Newton, e.g., with Copernicus and Kepler. Robert Hooke (1635-1703) came up with the idea of a decrease of gravitation with increasing distance from the Earth's surface. However, it was Newton who developed the theory of universal gravitation in 1666/67. While fleeing from Cambridge to Woolsthorpe in order to escape from the plague, he discovered during an exceedingly productive phase of his life that the gravitational acceleration of the Earth reaches up to and beyond the Moon and that the same gravitational acceleration causes a body to fall to the ground and the Moon not to escape from the Earth on a straight line. From the third Keplerian law he concluded that the force keeping a planet in its orbit must be inversely proportional to the square of the distance from the Sun to planet. It was, however, not until two decades after its discovery, that Newton's law of universal gravitation was published in section VIII of his famous *Mathematical Principles of Natural Philosophy* written in Latin.[39]

The fact that, reversely, the orbit of a body, affected by a central force inversely proportional to the square of the distance equals a conic, which means that the Keplerian laws result from Newton's law of universal gravitation, was proved by Johann Bernoulli.[2]

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* **Editor's Note**: See Werner Schulz, "Johannes Kepler and His Laws of Planetary Motion on the Occasion of the 350th Anniversary of the Day of Kepler's Death", in this volume.

At the same time when Newton discovered the law of universal gravitation, he laid the foundations of mechanics which are of fundamental importance to celestial mechanics as well as to all branches of classical mechanics. These conceptions found their expression in the three Newtonian axioms likewise published in the *Principia* (1687): the principle of inertia, the force law, and the principle of action and reaction. Incidentally, in one of Newton's lectures the remark is to be found that the third axiom supplies the basis for rocket propulsion in vacuo.[39]

The Keplerian laws represent only an approximate description of planetary orbits. For if one considers the Sun and a planet, it is not only the gravitational force of the Sun acting on the planet but vice versa, that of the planet exerting an influence on the Sun. The integration of the equations of motion for both bodies concerned leads to the Keplerian laws in a modified form: Each of the two bodies describes an ellipse around the common center of masses; equally, for each of the two bodies an area theorem is valid; and finally, the proportionality factor between the square of the period of revolution and the cube of the semimajor axis is not constant but depends on the masses of the Sun and the planet.

That Kepler was able to derive his idealized laws goes back, on the one hand, to the fact that the mass of a planet is small compared to the mass of the Sun, and that, on the other hand, the distances of the other celestial bodies which also exert gravitational forces on a planet are so large that it is possible to neglect the existence of the other planets in order to arrive at a good approximation when determining a planetary orbit, and to rest content with the equations of motion of the two-body problem. But Newton did already deal with the three-body problem as well, particularly for the case Sun-Earth-Moon.

Euler, Lagrange, Laplace

The names of the next two great scientists in the field of celestial mechanics are Leonhard Euler (1707-1783) and Joseph Lagrange (1736-1813).

Euler, whose scientific work covers nearly all branches of pure and applied mathematics, was the first to make use of analytical methods in the field of mechanics instead of applying geometrical synthetic procedures used hitherto, and to rely on Leibniz's form of calculus in order to represent the Newtonian dynamics of the mass point.[10] Euler published[11] a theory of the planetary and cometary motions dealing in particular with the problem of the calculation of disturbed orbits.

Space flight mechanics investigations of Euler from the 1760s are of interest, and deal with simple, special cases of the three-body problem and allow an analytical solution, namely, the problem of the two fixed centers[12,13] and the rectilinear motion of three bodies attracting each other according to the Newtonian gravitational law.[14]

When investigating the motion of a body subjected to the attracting forces of two fixed centers, it is of advantage that no centrifugal and Coriolis forces appear. In celestial mechanics the posing of this problem is only of academic interest. When considering, for instance, the motion of the Moon subjected to the influence of the

Earth and the Sun, it is inadmissible to suppose fixed centers because the Earth moves on significantly around the Sun during one orbit of the Moon around the Earth. For the motion of an artificial satellite, however, orbiting around the Earth within a short period it is quite possible to consider the geocentric position of the Moon or the Sun for a certain time as fixed, thus resulting in the Euler case of fixed centers for the three-body problem, Earth-Moon-satellite or Earth-Sun-satellite, respectively. For treating this problem, Euler introduced elliptic coordinates and a fictitious time. One is led to elliptic integrals. Following Euler, a number of famous mathematicians have also dealt with this problem, e.g., Lagrange, Legendre, Jacobi.

One should mention that the two-center problem can be used as a starting point for treating the planar restricted three-body problem. The case then lies as follows: Two bodies with finite masses describe circular (or more generally elliptic) orbits around their common mass center. To be found is the trajectory of a third body of negligible mass moving in the same plane under the influence of the two other bodies. By applying a coordinate system rotating with the two bodies of finite mass one is able to reduce the restricted three-body problem to the problem of the two fixed centers. In the well-known book by the Swedish astronomer Charlier[6] a perturbation theory of the planar restricted three-body problem on this basis is attempted. Samter[44] described in detail the procedures for the calculation. A recent representation of the problem making use of modern mathematical means was given by Arenstorf and Davidson.[4] Payne[41] developed perturbation theories of the two-fixed-center problem in three dimensions and in particular carried out numerical calculations for Earth-Moon trajectories. Finally, there should be taken into consideration two papers, the first one by Deprit[9] with a detailed classification of solutions of the two-fixed-center problem, the other one by Langebartel[30] where the author had in mind the application to the Apollo project.

Euler's[14] paper on the rectilinear motion of three bodies leads to the problem of collision of two bodies if the distance becomes zero and the velocity tends to infinity. Euler dealt in particular with the case in which the distances AB and BC of the three bodies A, B, C maintain a constant ratio. The transformation of the variables he used for the solution of this problem represents the first example for a regularization in more detail in a later paragraph of this paper.

A further major achievement of Euler consisted in developing the calculus of variations, the foundations of which had been laid by the brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748); among other things this is of importance for optimization problems in space flight mechanics. Closely related to Euler's name are, moreover, his gyroscope equations which were deduced in this mechanics[10] and which play an important role in flight mechanics.

Lagrange's contributions lie in developing the methods of analytical mechanics. His book *Mécanique analytique*[29] was written in Berlin where Lagrange carried out his research work at the Royal Prussian Academy of Sciences from 1766 to 1787 as successor to Euler, who had been director of the mathematics class since 1744 and who had moved to Saint Petersburg in 1766. With Lagrange, the use of geometrical figures plays no longer a part. He was able to write down immediately the equations furnishing the solution of extremum problems. However, he does not yet supply a mathematically rigorous motivation. Today, the Lagrangian equations are an indispensable tool for investigating the motion of a system of mass points under constraint.

It cannot be the task of this paper to deal explicitly with Lagrange's outstanding contributions to the theory of planetary motions and his fundamental results with regard to the three-body problem and, more generally, to the n-body problem which were the starting point for so many famous mathematicians in the 19th Century. However, it should be mentioned that Lagrange[28] succeeded in demonstrating that it is possible under certain special conditions to obtain rigorous solutions of the differential equations of the three-body problem. This is the case when the sides of the triangle formed by the three celestial bodies maintain constant ratios independent of time or, which is equivalent, when the angles of the triangle do not change in the course of time. Lagrange has shown the existence of solutions and indicated the initial conditions under which solutions exist. In this connection the five Lagrangian equilibrium points or libration centers play an important part. If a body describes an undisturbed conic (ellipse, parabola, hyperbola) around a second one there are five points in the orbital plane where a third body of negligible mass may be situated for all times. Three of these points usually called L_1, L_2, L_3 are situated on a straight line together with the two first bodies, the two other points, L_4 and L_5, form equilateral triangles with the two bodies. At the time of Lagrange this result seemed to be of no importance to astronomers. Since 1906, however, planetoids have been discovered, the so-called Trojans, which swing around the libration centers L_4 and L_5, respectively, of the Sun-Jupiter system. Thus the solutions of the three-body problem in the vicinity of the libration centers L_4 and L_5 become of interest. For space research the libration centers of the Sun-Earth and the Earth-Moon systems are of importance. It was proposed to carry out space flight missions to these centers[38,21] as one may expect to find there cosmic particles, the analysis of which should be enlightening.

One of astronomy's principal tasks consists of determining planetary and cometary orbits from observational data. From Newton stems a graphic method to determine a parabolic cometary orbit from three observations. Euler dealt with the problem of orbit determination and found starting points for an analytical treatment which, however, turned out not to be very suitable for practical calculations. The first practicable method of determining parabolic orbits, which - somewhat improved - has been used until today, was applied in 1797 by the physician and owner of a private observatory in Bremen, Wilhelm Olbers (1758-1840).[40] While for a parabolic orbit five unknown quantities are searched for, the determination of an elliptic orbit represents a more difficult task with six unknowns. In 1778 it was Lagrange and in 1780 Pierre Simon Laplace (1749-1827) who indicated a solution for this more complicated problem. Laplace is the author of five famous volumes on celestial mechanics[31] published in the course of a quarter of a century. A special study by Laplace concerning the activity sphere of a planet will be treated in a later paragraph of this paper.

Gauss

On 1 January 1801 Giuseppe Piazzi (1746-1826) in Palermo discovered a star of the eighth magnitude, the first of the minor planets orbiting between Mars and Jupiter. He was able to observe this star for 41 nights covering a distance of approximately 9 degrees and he named it Ceres. The then-known methods did not allow determination of the orbit with such few data available, considering that the orbit were an ellipse and not a circle or a parabola. At that time Carl Friedrich Gauss (1777-1855), then at age 24, had already occupied himself with problems of astronomy, in particular with the theory of lunar motion and the general problem of orbit determination. When Gauss learned about the data from Piazzi's observations which were published in the *Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde*, the first astronomical journal, edited since 1800 by Franz Xaver von Zach (1754-1832), he applied himself again to his former studies and defined a method to determine the elliptic elements from three complete observations (time, right ascension, declination). In December 1801 he announced the result of his calculations for Ceres in Zach's Journal, thus enabling Zach on 31 December 1801 at Gotha and Olbers on 1 January 1802 at Bremen to rediscover Ceres and thereby establishing a worldwide reputation for himself. After further perfection Gauss[15] published this method in his work on the theory of the motion of celestial bodies orbiting the Sun in conics.

The methods of orbit determination of Lagrange and Laplace, on the one hand, and of Gauss, on the other hand, differ fundamentally. In the first case differential equations are the starting point, thus setting up an initial value problem. In theory this way is very transparent but in practice there are difficulties resulting from numerical inaccuracies due to uncertainties in the initial values originating from the observations. Gauss based his procedure on the integrals of the two-body problem instead of the differential equations. Mathematically this means a boundary value problem dependent on time. The Gaussian method has proved very useful indeed.

Carrying out his calculations of the orbit of Ceres, Gauss made use of the method of least squares in order to compensate for random observational errors. He had derived this method already in 1794 when 17 years old, but he only published it in his above-mentioned book[15] and gave a second complete account on the theory of smoothing random errors and the method of least squares many years later[16], not because he had not seen its importance but because he assumed it to be obvious and had no particular interest in any claim of priority.

In this context, one should mention as second major field of application of calculations according to the least-square method the surveying of the Kingdom of Hanover done by Gauss. In order to overcome the difficulties in signaling the targets he invented the heliotrope, a tool consisting of a combination of a telescope and two vertically superposed plane mirrors, thereby allowing to turn the sunlight in any direction. Gauss took great pains constructing this tool and was considerably proud of his achievement. In a letter addressed to Olbers[45] he wrote:

"With a hundred mirrors, each of 16 square foot surface, and all used together, one would be able to send a fine heliotrope light to the Moon. What a pity that we are not in a position to send such an apparatus with a detachment of a hundred people and a few astronomers there that they might give us signs for effecting longitudinal determinations."

As with Lagrange and Laplace, it is also with Gauss impossible to comment in this paper on further achievements in the field of celestial mechanics which are of interest to space flight mechanics as well. In conclusion a quotation by the mathematician from Göttingen's Felix Klein (1849-1925) who says, in his lecture notes on the development of the mathematics in the 19th Century, that Gauss's *oeuvre* is characterized by the equilibrium between mathematical ingenuity, rigorosity of derivations, and a sense for practical application comprising careful observation and measurement as well as highly polished style.

Hamilton and Jacobi

The methods of perturbation calculations in celestial mechanics had been developed by Euler, Lagrange, Laplace and Gauss to an effect where they would suffice for all practical purposes, so that it seemed that, on the whole, this field was closed. One has to differentiate between the theory of general perturbations and that of special perturbations. In the case of general perturbations analytical results are obtained, e.g., according to Lagrange's method of the variation of constants, which allow statements on the character of the perturbations, i.e., secular, long-periodic, short-periodic. The case of special perturbations concerns investigations of concrete cases by means of numerical integration of the equations of motion. Incidentally, the term "special perturbation calculations" stems from Gauss.

However, in the 1830s and 1840s considerable progress was obtained in the development of methods for perturbation calculations in celestial mechanics due to results gained by William Rowan Hamilton (1805-1865) and Carl Gustav Jacob Jacobi (1804-1851) in their studies on general dynamic problems. Hamilton[18,19] developed a theory introducing into dynamics the "force function" which was generally made known by Gauss under the name "potential". Hamilton demonstrated that the integration of differential equations in mechanics can be reduced to the solution of two simultaneous partial differential equations. This did not seem to have much bearing in practice. It was only Jacobi[23] who recognized that the solution to be found had to satisfy only one of the two partial differential equations. Thereby the Hamilton-Jacobi theory supplied a mathematical way to formulate mechanical equations of motion, which essentially facilitates their integration. The system of differential equations is transformed into a so-called canonical form which is characterized by certain symmetrical qualities. Using the canonical variables, the "position coordinates" p_i and the "conjugate momentum coordinates" q_i, the canonical equations read:

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots, n),$$

H according to Jacobi meaning the Hamiltonian. In the case of conservative forces H represents the total energy of the system. Whereas Hamilton let himself be guided in his considerations by quite concrete concepts derived mainly from optics, Jacobi tended in his treatment of dynamical systems[20] to a strongly abstract-mathematical direction. Therefore, beginners have usually some difficulties in getting into the Hamilton-Jacobi theory. Being well acquainted with it is, however, frequently of use for space flight mechanics research.

Among the contributions of Jacobi there are also important findings regarding the three-body problem. The general three-body problem, i.e., with three finite masses and arbitrary initial conditions for the three bodies, is described by a system of differential equations of the eighteenth order. Lagrange had already known that one could indicate ten integrals of this system of equations: six from the theorem on the uniform motion of the center of gravity on a straight line, three from the theorem on conservation of angular momentum, and one from the theorem on conservation of energy, thus permitting the order of the system to be reduced to eight. The question is whether further integrals can be indicated. This has only been possible for special cases. Jacobi[22] indicated a further integral for the planar restricted three-body problem, i.e., with two finite masses on circular orbits and a negligible third mass, all moving in the same plane. Using a system of fixed coordinates this integral consists of terms of the forms of a kinetic energy, potential energies and an angular momentum. The sum of these terms is constant. This constant is called the Jacobian constant, the integral the Jacobian integral.

The restricted three-body problem is of importance for space flight mechanics. It may be used, for example, with a good approximation as starting point for flight trajectories from the Earth to the Moon provided that one takes approximately the Moon's orbit as trajectory plane for the spacecraft. The eccentricity of the Moon's orbit around the Earth is 0.055, so that this orbit may be considered as a circular one. The mass of the spacecraft is infinitesimal compared to those of the Earth and the Moon. Finally, one is justified in neglecting the influence of the Sun with regard to flights of only a few days' duration. The deviations from the assumptions of the restricted three-body problem can later on be worked out by perturbation calculations. For numerical calculations under the assumptions of the restricted three-body problem the Jacobian constant calculated for each step presents a valuable means of controlling the accuracy of the calculation.

In the case of the general three-body problem it is not possible to find an integral corresponding to the Jacobian one. A reduction of the order of the system of differential equations is, however, possible by eliminating a variable by means of a suitable transformation. Jacobi[24] indicated a way to do this which he defined as "elimination of the nodes".

Progress in Celestial Mechanics since the Middle of the 19th Century

In the second half of the 19th Century the theory of general perturbations of the motion of the planets, as it was initiated by Euler, Lagrange and Laplace, was accomplished in the way in which it is--apart from some smaller amendments and adaptations to modern developments in mathematics and computing techniques--valid until now by the director of the Paris observatory, Urbain Jean Joseph Le Verrier (Leverrier) (1811-1877), who published the results of general perturbations of his research in the memoirs of the Paris observatory.[32]

Before being appointed director of the Paris observatory, Le Verrier, incidentally, concluded in 1846 from perturbations observed with the planet Uranus that a planet not yet discovered must exist. When he indicated to the Berlin astronomer Johann Gottfried Galle (1812-1910) where presumably this unknown planet was located according to the perturbation calculations, Galle discovered, the very night when he received these data, the planet Neptune, almost exactly at the calculated position. This demonstrates the efficiency of the mathematical analysis of disturbed orbits.

Within this paper it is impossible to refer to the various and most interesting studies on the perturbations of the major and minor planets as well as on the motion of the Moon which date from the second half of the 19th and 20th Centuries.

The problems in celestial mechanics on which interest focused at the end of the 19th and the beginning of the 20th Century may be listed under the following keywords: nonexistence of integrals of a certain kind for the n-body problem, expansions into series of the solutions valid for arbitrary times, periodic solutions, asymptotic solutions for $t \to \infty$ and $t \to -\infty$, stability behavior, elimination of singularities in the equations of motion by a suitable transformation of the variables, i.e., research in celestial mechanics turned preferably toward the qualitative behavior of the solutions. In order to deal with such problems, auxiliary means of higher mathematics were required, above all theorems and methods of the theory of analytical functions and of differential geometry, which meant that progress in this wide field of research could be expected only from outstanding mathematicians. When the King of Sweden and Norway established a prize for the solution of the problem to find expansions into convergent series, valid for all times, for the coordinates of n bodies attracting each other according to Newton's law, this prize was awarded in 1889 to the French mathematician Henri Poincaré (1854-1912) who had entered the competition with a comprehensive treatise on the three-body problem and the differential equations of the dynamics of mass points. Though Poincaré (1890) did not solve the actual problem, his paper contained so many new ideas which proved to be very fruitful afterwards that the award was certainly justified. Later on Poincaré published two major studies in three volumes each which is still a source for new ideas: the first outlines the new methods;[42] the second contains Poincaré lectures on celestial mechanics.[43]

The difficulties to find expansions into series for the solutions of the n-body problem valid for all times were, among others, due to the fact that one was unable to indicate restrictions for the initial conditions which would have led to the conclusion of the impossibility of collisions of two bodies. For the three-body problem the Finnish astronomer Karl Frithiof Sundman (1873-1949) succeeded in 1913 in obtaining formal solutions by convergent expansions into series and their analytic continuation by choosing, instead of time, another independent variable so that the functions remain regular also in the case of a collision of two bodies.[47] For practical applications, however, this solution is of no use, as the convergence is only very weak.

The studies dealing with the problems of celestial mechanics indicated by the above-mentioned keywords are in particular of interest to mathematicians. For engineers involved in practical astrodynamics they are of minor concern, excepting the problem of eliminating the existing singularities in the equations of motion by means of regularization. As already mentioned, the beginnings of dealing with this problem date back to Euler,[13,14] and Sundman[47] too has to be mentioned in this respect. Prior to Sundman, already toward the end of the 19th Century, Burrau[4] and Thiele[49] developed regularization procedures. A major progress was then achieved by the Italian Tonio Levi-Civita (1873-1941) who started about 1903 to study the problem of collision and the regularization maintaining the canonical form of the system of differential equations and who published over a period of two decades a number of valuable papers out of which only the first ones shall be mentioned here.[33,34,35] In a lecture presented in 1923 in Barcelona, Levi-Civita described in detail the transformation applied by him for the coordinates and the time for the treatment of the planar restricted three-body problem, and he also explained his wasted efforts with respect to a regularization of the three-dimensional case.[36]

A slightly different transformation from that of Levi-Civita was applied by the American George David Birkhoff (1884-1944) for the regularization of the restricted three-body problem. With Birkhoff's[3] transformation the origin of the system of coordinates is not in the center of gravity of the two bodies of finite mass but in the geometric center.

APPLICATIONS OF RESULTS AND METHODS OF CELESTIAL MECHANICS TO SPACE FLIGHT MECHANICS

Four examples shall now serve to indicate briefly how findings in celestial mechanics were applied to problems of space flight mechanics. For a comprehensive survey with a list of references one should refer to a paper by the author: "Bemerkungen zur Entwicklung der Raumflugmechanik; von der Himmelsmechanik zur Raumflugmechanik." DGLR Mitteilung 78/01 Köln 1978, pp.206/1-44. (Ed).

Optimum Transfer Trajectories

Kepler's laws were first applied to space flight missions in connection with theoretical studies of the German engineer Walter Hohmann (1880-1945). During World War I Hohmann investigated how to reach other planets, in particular Venus, by means of rocket vehicles. The well-known result today was that, after overcoming the Earth's attraction, the appropriate trajectory is the Keplerian semi-ellipse flown without thrust which leaves the supposedly circular orbit of the Earth around the Sun tangentially and enters into the coplanar circular orbit of Venus around the Sun tangentially. The transfer from the initial circular orbit into the ellipse and the transfer from the ellipse into the second circular orbit are effected by instantaneous thrust impulses. This king of transfer trajectories is now called the Hohmann transfer.

Hohmann also indicated how Earth and Venus must be related to each other so that the spacecraft when entering the Venus' orbit then reaches the planet, and how often Venus would have to be orbited before starting on the return flight in order to reach the Earth on a second semi-ellipse. According to the third Keplerian law, the time for the flight from Earth to Venus on the semi-ellipse amounts to 146 days; the length of time for which the spacecraft was to stay on orbits around Venus was determined by Hohmann as 464 days, thus resulting in a total flight time of 756 days, 146 days for the return flight to the Earth included.

It is interesting that Hohmann had in mind a second possibility, to fly from the Earth to Venus and back for which he stipulated a flight time of 1.5 years = 547.5 days. This trajectory consists of three semi-ellipses. As in the aforementioned case, the first semi-ellipse allows for the transfer from the Earth's orbit to the Venus' orbit. When arriving there the spacecraft gets a second thrust impulse causing it to enter into an ellipse reaching beyond the Earth's orbit. In the aphelion of this trajectory a further thrust impulse is given which ensures that the third semi-ellipse enters the Earth's orbit tangentially. The distance of the aphelion of the second ellipse is chosen to the effect that the time for passing through all of the three semi-ellipses equals the time for one and a half orbits of the Earth around the Sun. Thus, the spacecraft really reaches the Earth.[20]

Generalizations of the Hohmann transfer have been undertaken in various respects. The most evident ones concern two-impulse transfers between any coplanar conics. The next step leads to transfers between non-coplanar initial and final orbits. One has to differentiate between two categories of problems depending on whether the transfer time is open or fixed. Finally, boundary conditions for the start and the end of the transfer trajectory may be given which concern the position vector, the velocity vector or the flight path angle. Usually a transfer trajectory with a minimum demand in change of velocity (= fuel consumption) is required.

Another complex of questions concerns the optimal number of impulses for the transfer. There exist, for example, three-impulse transfers between two coplanar circular orbits, so-called bi-elliptic transfers, which are more economical than the Hohmann transfer.

Thus, a number of questions are raised which are not only of interest for the theorist but also for the space flight engineer. The number of investigations which have been carried out in this field since the beginning of the fifties is correspondingly large. A very comprehensive survey of the papers published until 1968 on impulsive transfer was given by Gobetz and Doll[17] who listed 316 references. Since then many more papers have been published and there are still questions on this subject which are pending.

In the simplest cases a mathematical treatment with calculus is possible. In more complicated cases for analytical investigations, not only the calculus of variations in its classical form is used but also theories with fewer restrictive conditions, such as Pontryagin's maximum principle. Because not in every case solutions in closed form exist, there are numerous numerical investigations where, for instance, methods of steepest descent or nonlinear programming have proved to be suitable.

Here it may suffice to mention only a few papers originating from ONERA in Paris[7,37] and from the Department of Aerospace Engineering Sciences of the University of Colorado in Boulder, Colorado[5,8] where new ideas were developed. Busemann presented his concept at the 9th Ludwig Prandtl Memorial Lecture* organized in DGLR and GAMM† in Vienna. It concerns the use of a phase space for the investigation of transfers between coplanar ellipses. Each ellipse is represented by a point of the phase space. The impulse divided by the mass of the spacecraft defines the "distance" between neighboring points. Having then determined the displacement vector which corresponds to the unit impulse for each point toward every direction, one has to seek the "shortest" connections, the "geodesics" between any two points. It results that the tensor of all displacement vectors can be concave, which implies that one has to form its hull in order to obtain the metric tensor. This seemingly rather abstract construction can be well visualized geometrically and is suitable for finding optimum transfer orbits.

Laplace's Activity Sphere

In volume IV (pp.216-228) of his *Traité de mécanique céleste*[31] Laplace divided the orbit of a comet swingingby the planet Jupiter into two parts. In the vicinity of Jupiter the gravitational force of this planet predominates over that of the Sun; the orbit of the comet there may be taken as a conic (hyperbola) having Jupiter as central body disturbed by the Sun, i.e., as a conic in a jovicentric coordinate system. Farther away from Jupiter the orbit must be considered as a conic in a heliocentric coordinate system. In order to define the geometrical locus where it is most expedient to effect the change from the one coordinate system to the other, one determines for the points of the space the quotients from the acceleration by the disturbing body to the acceleration by the central body. If for a point the quotient with Jupiter as central body is smaller than the quotient with the Sun as central body, this point lies within the activity domain of Jupiter with respect to the Sun. If both quotients are equal, the point lies on the boundary of the activity domain. The activity domain is approximately a sphere called the activity sphere of the planet. The concept of the activity sphere simplifies the integration of the equations of motion by allowing reduction of the three-body problem to two-body problems.

Of the same nature as the three-body problem Sun-Jupiter-comet are the three-body problems: Sun-planet-spacecraft and Earth-Moon-spacecraft. In all of these cases the mass of the third body is negligible compared to those of the two

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† **Editor's Note**: GAMM: Gesellschaft für Angewandte Mathematik und Mechanik.

others. Thus in the case of interplanetary space flights it is possible to change the rate and direction of the velocity vector of a spacecraft without thrust by utilizing the gravitational fields of planets and planetary moons and to apply the concept of the activity sphere for feasibility studies. Noteworthy examples in this respect are the flights of *Pioneer 10* and *11* as well as those of *Voyager 1* and *2* with swingbys past Jupiter in order to continue the flight to Saturn on extra-ecliptical trajectories and possibly with swingbys past Saturn permitting the space probes to escape from the Sun's system.

Disturbed Keplerian Orbits

The general perturbation theories of first order as well as of higher orders which have been developed since the 18th Century for closer studies of the planetary motions are quite applicable to motions of artificial Earth's satellites and space probes enabling one to derive laws concerning the influence of disturbing forces. Moreover, there exist modifications and further development of these theories with regard to the needs of space flight mechanics. As disturbing forces there exist not only the gravitational forces effected by the Sun, the Moon and the planets, but more important than those in certain fields are the effects of the atmospheric drag and the deviations of the Earth's shape from a sphere with homogeneous mass distribution. Under certain circumstances the solar radiation pressure can exert a noticeable influence, in particular in case of satellites with large surface and small mass.

In case of low-altitude orbits the drag presents the most important perturbation of the Keplerian orbit. What makes it difficult to predict the trajectory exactly is the fact that the drag varies considerably with time and locus. Based on a standard atmosphere with average amounts for air density, however, general statements concerning the effects of the drag on the elements of the Keplerian ellipse are possible. It is equally possible to take into consideration the oblateness of the Earth at the poles by applying the relevant potential of gravitation in the equations of motion. The drag being no conservative force the energy law does not apply any more. The gravitational force is no longer, as in the case of a spheric Earth, a central force and thus the area law for the orbital plane does not apply any more.

A differentiation between the effects of drag and of the Earth's oblateness is not easily done, but can be achieved to a certain point because both these perturbations differ in character. By applying the general perturbation theories one comes to the conclusion that the drag causes secular changes of the semi-major axis and the eccentricity of the satellite orbit, whereas these two elements change periodically under the influence of the Earth's oblateness. On the other hand, the Earth's oblateness causes secular changes of the right ascension of the node and the argument of perigee, whereas these elements change periodically under the drag influence. The Sun and the Moon cause only minor periodical changes for all orbital elements.

It may suffice to mention a few noteworthy results of the effects of drag and the Earth's oblateness on satellite orbits: The semimajor axis and the eccentricity become increasingly smaller under the influence of drag, the orbit becomes increasingly circular. The period of the orbit decreases. With elliptic orbits the drag is mostly considerably greater in the vicinity of the perigee than elsewhere. The distance of the apogee is then reduced about twice as much as the semimajor axis while the distance of the perigee changes only slightly. The velocity at perigee becomes smaller, that at apogee larger. In case of orbits with small eccentricity the increase of the velocity at apogee amounts to three times as much as the decrease of the velocity at perigee. The Earth's oblateness affects the ascending node and the perigee as follows: If the satellite crosses the equator with an ascending node with a component directed toward the east (west) the ascending node moves towards the west (east). For inclinations $i < 63.4^o$ there results a turning of the orbital ellipse (i.e. the semimajor axis) in the same sense as the satellite moves. Each point of the orbit becomes perigee in course of time. For $i > 63.4^o$ the ellipse turns in the opposite sense. For $i = 63.4^o$ the perigee stays continuously over the same parallel of latitude. The atmosphere rotates together with the Earth. This rotation causes a small secular change of inclination. In addition small periodical changes of inclination are caused by the oblateness of the Earth. The Moon and the Sun cause only slight periodical changes for all orbital elements.

In case of satellites with great eccentricities and periods of several days the perturbations caused by the Sun and the Moon must be taken into account carefully. The changes of the orbital elements depend on the positions of these celestial bodies in relation to the satellite changing according to time. Whereas the life time of satellites with low altitude orbits depends solely on the drag, it depends in case of high-eccentricity orbits on the right ascension of the node at the beginning and may vary considerably.

Regularization and Linearization of the Equations of Motion

As mentioned, ways and means had been found at an early date to abolish the singularity in the differential equations of the two- and the planar three-body problem which exists if the distance r between two bodies becomes zero, i.e., in case of collision. For this one introduces instead of the time t a pseudotime s according to $r\,ds = dt$ and effects a transformation of the coordinates. Levi-Civita chose a conformal mapping of the orbital plane $z = x + iy$ on a parametric plane $w = u + iv$ by the transformation $z = w^2$, i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} .$$

Levi-Civita's attempts to generalize this procedure to three-dimensional space remained unsuccessful. It was only in 1964 that the Finnish astronomer Kustaanheimo achieved this by applying spinors. In 1965 he published jointly with the Swiss mathematician Stiefel a very elegant solution which indicated at the same time the reason for the former failures, because in order to solve the three-dimensional problem one has to approach it via a four-dimensional space. The transformation of the spatial coordinates x, y, z on a four-dimensional parametric space u, v, w, q, which leads to the regularization, is:

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} u & -v & -w & q \\ v & u & -q & -w \\ w & q & u & v \\ q & -w & v & -u \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ q \end{bmatrix} .$$

If one puts $w = 0$, $q = 0$ one obtains the transformation of Levi-Civita. According to a theorem by Hurwitz of the year 1898 on the composition of quadratic forms of n variables matrices as used here in the cases $n = 2$ and $n = 4$ exist too for $n = 8$ but in no other cases, in particular not for $n = 3$.

If one effects the transformation, the differential equations for the Keplerian motion become linear differential equations with constant coefficients which are regular for $r = 0$. For the planar Keplerian motion of the two-body problem the transformed equations are the equations of a harmonic oscillation.

The application of the Levi-Civita transformation and the so-called (after its authors) KS-transformation for the regularization and linearization of the equations of motion is not only of importance for collision orbits but also if a spacecraft comes very close to a celestial body. The exactness of numerical integrations of Keplerian orbits, including disturbed ones, can be greatly improved by applying regularized equations. Significant studies in this field have been carried out in particular in the Institute of Applied Mathematics of the ETH Zürich (Prof. Stiefel). Detailed representations of the theory of regularization can be found in the books of Szebehely *Theory of Orbits*[48] and Stiefel/Scheifele *Linear and Regular Celestial Mechanics*.[46]

CONCLUSION

This paper attempted to outline how solutions to problems of celestial mechanics arrived at over the last few centuries have proved useful for space flight mechanics, and that the same also holds true with regard to mathematical methods the applicability of which was originally by no means obvious. This last point, however, is a fact that has been established for a very long time. When the Greek geometer Appolonius compiled around 200 B.C. a systematic representation of conics in his eight-volume opus *Conica*, it was nobody's guess that Kepler would draw on these findings with regard to the orbits of the planets.

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