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Chapter 18

Goddard's 85 Years Optimal Ascent Problem Finally Solved*

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Abstract

Robert H. Goddard was the first to observe, by physical reasoning, that, if a rational ascent speed policy is followed, a rocket vehicle might reach a given altitude with a minimal starting mass, meaning with the least possible fuel consumption. He published his observations in 1919, suggesting a variational approach could be used to find the solution, but gave none. This is the Goddard problem of rocket ascent. Only four years later, Hermann Oberth had independently published a similar discussion on the optimal atmospheric ascent and gave in 1929 the first, approximate solution of the problem, strikingly resembling the later, more evolved ones. The well-known professor in mathematics Georg Hamel formulated, in the meantime, (G. Hamel, "Eine mit dem Rakete zusamenhängende Aufgabe der Variationsrechnung," *ZFW,* December 1927) the Goddard problem in strong variational terms that stated the basis for all eventual developments on the subject, but still with no explicit solution. Despite a number of earlier trials, H. S. Tsien and R. Evans are the first to find a partial, variational solution of the Goddard problem in 1951, in a very beautiful work. A flow of developments in atmospheric ascent optimization then emerged with hundreds of high

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quality papers, including the main ones by I. Gudju, G. Leitmann, A. Miele, and others. By continuous variational methods the solution is found that consists of an impulsive start, followed by an accommodated sustainer flight, and the injection to coast up to the peak altitude. But in all these solutions, the basic discontinuity in the equations of motion at burn-off is fortuitously neglected. Salutary are the impressive solutions of Bulirsch's team in Munich, which emphasize the challenge of discontinuities and give some answers. Despite these considerable attempts, pure numerical solutions are currently only adopted to optimize, in part, the vehicles' ascent, as the classical calculus of variations proves inadequate for this discontinuous integrand problem. In fact, the entire amount of variational work does not respond to the real problem of rocket ascent and we realize that up to recent time no complete and documented answer to the 85-year-old Goddard problem seemed found. We try to follow in this paper the history of those tremendous findings, which had only found very recently an unexpected answer.

First Formulation in 1919

In his famous pioneering paper[1] from 1919, Robert H. Goddard first observed that the atmospheric ascent of rocket vehicles is clearly subjected to the optimal thrust control. We reproduce this first formulation from the classical work of Goddard:[1]

> Suppose that, at some altitude on the ascending path of a rocket, the ascent speed will be very high. Then the force of air drag, proportional to the squared speed, will also be very high. On the other hand, at very small climbing speed, during a long time period a thrust is needed to compensate for the force of gravity. In both cases, the required propellant reserve will be very high. It follows that the climbing speed must have a precisely determined value at each point in space.

Goddard did not solve this problem, due to his "improper mathematical means," as Hamel comments later in 1927. The problem Goddard addresses belongs, in fact, to the calculus of variations and, much more than this, to the yet unsolved category of discontinuous integrand problems.

The mandatory impulsive start of the vehicle was, however, revealed. This difficult point remains in flight optimization and is continuously pursued by the rocket engine designers who had consequently introduced the famous launch thrust augmenters. *No word yet of a burnout discontinuity*, however.

Figure 18–1: Robert H. Goddard.

Approximate Solution of Oberth

In Romania[2] in 1929, Hermann Oberth had developed the idea of the optimal thrust rule by a local differential scheme. He found his own answer in the form of equality between drag and weight during the powered flight. This approach means that in each individual point of the ascent trajectory a maximal gain in altitude dh and impulse $d(mv)$ in respect to the flight speed v are intended, with the least momentary mass expense dm. This criterion proves different from that of Goddard[15] and in fact is a mini-max criterion. In order to arrive at simple formulae, constant values were considered for the air density, the specific impulse of the engine, and the drag coefficient while doing differentiation. This is the usual method of frozen coefficients.[14] All are common, acceptable hypotheses.

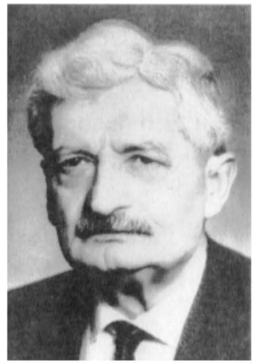

Figure 18–2: Hermann Oberth.

The equation in differentials of the powered, rectilinear motion was accordingly written in the usual form,

$$m \cdot dv + c \cdot dm + Q \cdot dt = 0, \qquad (1)$$

where c is the specific impulse of the engine and Q the opposing forces, viz. the sum of gravity and drag,

$$Q \equiv mg \sin\alpha + R. \qquad (2)$$

Oberth had considered now the variation of the infinitesimal impulse $d(mv)$ induced by a velocity increment

$$\frac{\delta[d(mv)]}{\delta v} = \frac{\delta}{\delta v}(m dv) = 0, \qquad (3)$$

where $\frac{\delta}{\delta v} dm = 0$ as far as the mass variation goes through a minimum. The term mdv is extracted from the equation of motion. Then the extremum condition becomes,

$$\frac{\delta}{\delta v}(m dv) = -\frac{\delta}{\delta v}(c dm) - \frac{\delta}{\delta v}\left[\frac{Q}{v}\frac{dh}{\sin\alpha}\right] = 0.$$

Again the differential principle says that the variation of altitude goes through zero and consequently the variational condition of Oberth gets the remarkably simple form

$$\frac{\delta}{\delta v}\left[\frac{Q}{v}\right]=0. \tag{4}$$

For a simple quadratic drag law with constant c_D and under the assumptions of constant gravity field,

$$Q=\frac{\rho}{2}Sc_D v^2+mg\sin\alpha, \tag{5}$$

$$\frac{\delta}{\delta v}\left(\frac{\rho}{2}Sc_D v+\frac{mg\sin\alpha}{v}\right)\equiv\frac{\rho}{2}Sc_D-\frac{mg\sin\alpha}{v^2}=0$$

the intuitive supposition of Oberth for an optimal flight is proved,

$$R=G=\frac{Q}{2}. \tag{6}$$

Thus the most favorable speed of motion in the sense of Oberth, is given simply by the condition (6),

$$v=\sqrt{\frac{mg\sin\alpha}{\frac{\rho}{2}Sc_D}}. \tag{7}$$

Oberth further gives his typically instructive improvement, when the equation (7) is used to determine the derivative of the mass in respect to the speed. The mass is thus eliminated from the equation of motion (1), where the very acceptable assumption of an exponential atmosphere is used

$$\rho(h)=\exp(-\alpha h)$$

and thus a differential equation of the optimal motion results,

$$\frac{dv}{dt}=\frac{\alpha v-2g/c}{1/c+2/v}. \tag{7}$$

We may count it like an equivalent of an Euler equation. Two-fold successive integration renders the equations of optimal atmospheric ascent after Oberth,

$$z-z_A=(1+\beta)\ln\frac{x-2\beta}{x_A-2\beta}-\ln\frac{x}{x_A}, \tag{8}$$

$$y-y_A=x-x_A+2(1+\beta)\ln\frac{x-2\beta}{x_A-2\beta}. \tag{9}$$

The Tsien notation[6] $\beta\equiv g/(\alpha c^2)$ is used and the nondimensional speed, altitude, and time are $x=v/c$, $y=\alpha s$ and $z=gt/c$. Formula (9) is a good improvement of the speed (7) along the accommodated trajectory.

Equations (8) and (9) of Oberth visibly resemble the future solutions of Tsien, based on definitely strong variational principles. Obviously, this first response to the problem of flight optimization is valuable as a first closed, numerical set of formulae that can be used in the computations.

Although they seem based on apparently doubtable mathematical manipulations, the ultimate word comes from the numerical results. The original book contains minor numerical data only, the author's main concern remaining the theoretical solution.

As an intimate of Oberth's works, the author programmed the above formula into a computational code to compare in detail, for the first time as far as we know, today's variational solutions and Oberth's. The comparative research[14] was published in 1989 and sent to Hermann Oberth for his consideration (Figure 18–2). This comparison is presented further and shows an unexpectedly good agreement between the old one and the quite exact solutions of today.

Formula (7) shows that the optimal velocity never equals zero, meaning that the optimal vehicle must, from the beginning, start impulsively to acquire the condition of best ascent. This confirms the intuitive assessment of Goddard about the impulsive start, as Oberth himself is observing in the second[3] of his books from 1929.

The question remains why Oberth's solution is not exact? The optimality principle itself answers: the simultaneous propellant expense minimization and altitude maximization were considered, ending in over-conditioning of the equations that describe the vehicle behavior. In fact, the Goddard problem is not a mini-max, but a simple one: either minimal mass for given height or maximal height for given mass as well.

Still no word of the discontinuity of flight equations at engine's burnout.

Variational Formulation, Hamel 1927

Well-known especially for his works in theoretical mechanics,[5] the former professor in mathematics at the German university in Karlsruhe, later in Brün, Aachen, and finally Berlin, Georg K. W. Hamel (1877–1954) had been attracted by the "yet unsolved variations problem" of Goddard.

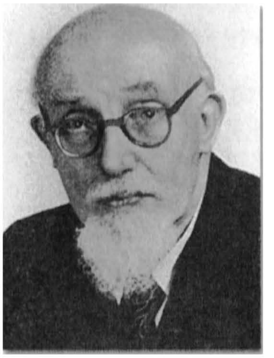

Figure 18–3: Professor Georg K. W. Hamel.

His paper[4] from ZAMM in December 1927 is just two pages long, but nonetheless very dense. From the equation of rectilinear ascending motion in the atmosphere

$$M\frac{du}{dt}+C\frac{dM}{dt}+W(s,u)+Mg=0 \quad (10)$$

the famous functional of the initial mass of the rocket vehicle M_a is rendered,

$$M_a \equiv \int_0^{t_e} f(u,s,t)\,dt + F(u_e,t_e) = \min. \quad (11)$$

resulting in a variational Euler type problem, in which no initial conditions are required, as they are contained in the functional itself. The end-point $\{s_e,t_e\}$ (engine burnout) is free, except that it must fulfill the connection or transversality condition with the free climb subarc. This states that an ordinary variational problem appears, with an Euler equation and regular extremals associated.

Hamel shows that the functional (11) presents a real minimum because the following condition is always fulfilled, namely the second order derivative

$$\frac{\partial^2 f}{\partial u^2} > 0, \quad (12)$$

as long as the drag (Widerstand) is always

$$W \geq 0,\ \frac{\partial W}{\partial u} \geq 0,\ \frac{\partial^2 W}{\partial u^2} > 0.$$

Hamel also shows, that difficulties begin when the end-point $\{s_e,u_e\}$ is varied along the coasting subarc. This sounds like Georg Hamel could not have obtained closed analytical formulae, first derived by Tsien, only 20 years later. In the attempt to surpass the obstacle, he shows that for the Jacobian of the second order

$$\frac{\partial\left(\frac{\partial M}{\partial s_e}, -\frac{\partial M}{\partial t_e}\right)}{\partial(s_e,-t_e)}, \quad (13)$$

any variation the discriminant is null,

$$\left[\frac{\partial^2 M}{\partial {s_e}^2}\frac{\partial^2 M}{\partial {t_e}^2} - \frac{\partial^2 M}{\partial s_e\,\partial t_e}\right]_0 = 0, \quad (14)$$

so that the admissible sharp domain of M_a arrives at $\{s_e,u_e\}$, where M_a is smaller than in $\{s_0,u_0\}$. Hamel enters also the question of an end-point mass intake by the engine, but considers it like physically impossible. Ross[8] admits it as a theoretical point in 1958.

For the first time Hamel points out *the discontinuity of air drag at burnout, completely out of reach for the classical calculus of variations* and visibly displays, in an extremely lucid manner, the basic hypothesis that the engine exhausts do NOT influence the drag. Basically, the exhausts affect the drag by 30 percent!

The totality of later works on the subject ignores this and follows the Hamel assumption of 1927.

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11. Über eine mit dem Problem der Rakete zusammenhängende Aufgabe der Variationsrechnung.

Von G. HAMEL in Berlin.

Läßt man einen starren Körper von der augenblicklichen Masse M unter der Wirkung der Schwere, des Luftwiderstandes W und der Reaktionswirkung ausströmender Materie (also eine Rakete) in die Höhe steigen, so erhält man aus dem Newtonschen Grundgesetz der Mechanik und dem Gesetz der Massenerhaltung die Differentialgleichung

$$M\frac{du}{dt} + C\frac{dM}{dt} + W(s,u) + Mg = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Dabei ist s der Weg, t die Zeit, $u = \frac{ds}{dt}$ die Geschwindigkeit, C die relative Ausströmgeschwindigkeit der Raketenfüllung.

Es sind folgende Vernachlässigungen gemacht: 1. In Anbetracht dessen, daß man nur Höhen von 100 bis 200 km erreichen will, ist g konstant genommen. 2. In W ist der Einfluß der ausströmenden Masse fortgelassen. 3. Die Aenderung des Impulses im Innern der Rakete durch Fortschreiten der Brennfläche (oder Aehnliches) ist als belanglos weggelassen. 4. Von der Erdrotation ist abgesehen.

M_e sei die Endmasse, M_α die Anfangsmasse. Gegeben seien: M_e, die gesamte Steighöhe h, ferner die Anfangsgeschwindigkeit u_α zur Zeit $t_\alpha = 0$, $s_\alpha = 0$, die konstante Ausströmgeschwindigkeit C. Gefragt ist nach dem Minimum von M_α, der Anfangsmasse.

Diese Aufgabe wurde von Goddard[1]) gestellt und zu lösen versucht, aber mit anfechtbaren mathematischen Hilfsmitteln. Sie soll hier mit den Mitteln der Variationsrechnung gelöst werden.

Aufgefaßt als lineare Differentialgleichung in M, läst sich Gl. (1) integrieren und nach Einsetzen der Endwerte nach M_α auflösen:

$$M_\alpha = e^{-\frac{u_\alpha}{c}} \int_0^{t_e} \frac{1}{c}\, W(s,u)\cdot e^{\frac{u}{c}+\frac{gt}{c}}\, dt + M_e\, e^{-\frac{u_\alpha}{c}+\frac{u_e}{c}+\frac{gt_e}{c}}$$

Figure 18–4: Facsimile of Hamel's paper.[4]

The expertise of Professor Hamel in the calculus of variations proved essential. In fact however, he did not solve the problem of Goddard completely. He gives no detail about the formulae that render the extremal and the limit conditions of flight, although the realistic value of 1000÷1100 m/s for the speed of accommodation u_0 is advanced. More developments were further required to arrive at design formulae. The conclusion up to here is that Hamel, like all those who followed, explicitly avoids the burnout discontinuity, which subsisted as the major "hindernis" against a complete variational solution.

An Important Step: Tsien 1951

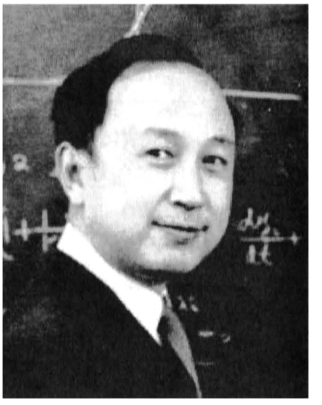

Figure 18–5: H. S. Tsien (born 1909).

Attracted by the same rocket ascent problem, the young doctor and recognized scientist in rocketry of Caltech Hsue-shen Tsien, born in 1909 in China, answered more exactly the variational formulation of Hamel. In 1951 H. S. Tsien published an important paper,[6] containing all missing formulae that render the injection point for coasting, the flight altitude, time, acceleration, and the mass variation along the extremal arc of Hamel's problem, all in closed, analytical form.

To arrive at a practical result, Tsien followed step-by-step the entire inner cuisine of variational calculus for Euler-type cases with a free final end point, in connection to a given curve, in this case to the unpowered coast arc. This article is remarkably clear and shows a lot of genuine mathematical findings, related to the physical interpretation of the Lagrange's deviameter.

It is shown for instance that the role of the variational parameter ε of Lagrange in its quasi-extremal $s(t,\varepsilon)$,

$$s(t,\varepsilon) \equiv \sigma(t) + \varepsilon \cdot \eta(t) \tag{15}$$

is the extension of the burning time of the engine t_B, besides its basic one of varying the deviated extremal $s(t,\varepsilon)$ off the extremal $\sigma(t)$ as a deviator (Figure 18–6).

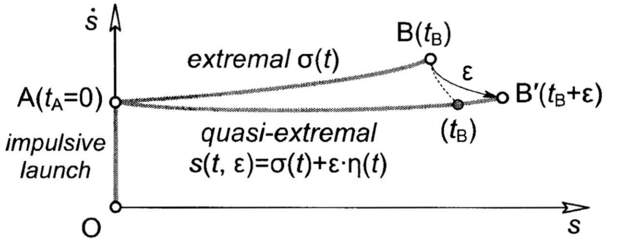

Figure 18–6: The effect of the Lagrange deviator ε.

Although apparently an ordinary fact, the emphasis on the role and meaning of Lagrange deviators opens the avenue toward developments in basically discontinuous problems, beyond the limits of the calculus of variations. It was perhaps too early in 1951 for such a breakthrough, but this extension did not appear even to the end of the 20th century.

Ample numerical results are detailed by Tsien. The continuity of derivatives at cut-off is unconditionally accepted and the burnout locus meets this assumption. It is clear that any discontinuity would have induced disastrous effects. The Hamel hypothesis was not questioned. *This work had solved the Goddard problem without a full answer yet.*

Optimization Efficiency: Leitman 1957

Continuing the series of analytical solutions of the rocket ascent problem, the mathematician George Leitman developed a stream of works on the subject, with similar conclusions as Tsien, and two major improvements. Leitman uses the Bliss first integral of Euler equations in the Hamel problem to derive the accommodation equation, already[6] deduced by direct means. He shows in this manner that the known but controversial impulsive start in the optimal rocket ascent is intrinsically required. The condition at cut-off in the form given by Leitman proves to play a central role in optimization problems,

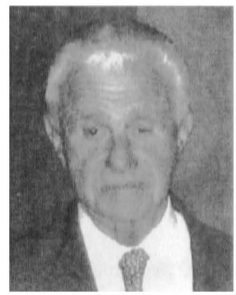

Figure 18–7: George Leitman.

$$\dot{\sigma}_B\left[\left(\frac{\partial R}{\partial \dot{\sigma}}\right)_B+\frac{R_B}{c}\right]-R_B-\mu_B g=0\,. \tag{16}$$

Here the apparent acceleration of drag, or rather retardation, is denoted by R and usually the mass ratio at the very cut-off is $\mu_B=1$.

The most important contribution of Leitman is the comparison between the extremal flight and the best constant flow-rate ascent.[7] The small gain in altitude that resulted from the variational thrust law, published as 11 percent only (6 percent actually), was discouraging. Responsible for the small value is the high penetrability of the rocket used in the example ($3\cdot10^5$ m), which drastically reduces the optimization effects. Nothing was also shown regarding the effect of a variable specific impulse $c(h)$ of the engine with altitude (altitude characteristic) and regarding the sharp dependence of the drag coefficient of speed, which was still considered as constant. *And still nothing about the discontinuity at cut-off.*

Developments of Miele

Based on the solid support of the methods of Bliss in the calculus of variations, Angelo Miele develops in a lifelong series of works of different approaches to flight optimization problems. His original, non-variational method[10] of optimization for linear differential equations of motion, based on Green integrals, is elegant, as are all of his works on the subject. For example, in the whole variational method after Lagrange, Miele shows that all the variables, derivatives, and functions involved in the integrand, as defined by the functional, must actually fit a raised degree of continuity along the entire range. Specifically, all variables and the integrand itself of the functional,

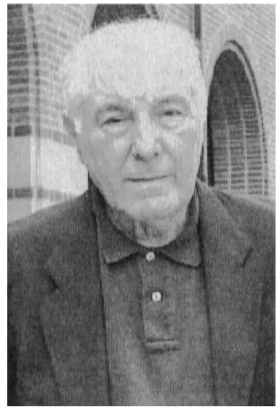

Figure 18–8: Angelo Miele.

$$I(C) = \int_{x_0}^{x_f} \breve{f}(x,y,\dot{y})\,dx \to \min_{u(x)} \tag{17}$$

must possess continuous derivatives up to the fourth order on the entire range,[9]

$$f \in c_4(x,y,\dot{y}). \tag{18}$$

The upper point on a variable denotes differentiation in respect to x and the value $I(C)$ is computed along

$$C_i = \{y_i(x) \mid i = 1,n\}, \tag{19}$$

called admissible arc if it complies itself to the requirement of continuity,

$$C_i \in c_1(x). \tag{20}$$

For this problem a series of questions regarding the staging processes, the aerodynamic properties and cut-off/start-up of the rocket engines, all related to discontinuities, are omitted. The valuable solutions of Miele for some rocket and aircraft optimal problems are still based on *full continuity properties of integrand.*

German Advances at Munich

The chair of mathematics at Technische Universität München (TUM) in Munich developed, more recently, a long series of numerical computations for optimal flight control, with respect to the optimization of the atmospheric ascent of projected dual-stage Sänger space system. Bulirsch,[11] Oberle,[12] Pesch,[13] Chudej,[11] and others extensively dealt with staging discontinuities problem.

Figure 18–9: Roland Bulirsche.

They showed that this class of variational problems involve inequality constraints in the control variables[13] rather than in the whole functional itself,

$$S(x(t)) \le 0, \quad S : \Re^n \to \Re^{\bar{l}}. \tag{21}$$

These complex difficulties were solved without changes in the mathematical background of the calculus of variations. Still the most unavoidable discontinuity of the blunt decrease in drag at power cut-off and increase at restart of upper stages *remains unsolved.*

The Severe Discontinuity

The exact location of the severe discontinuity is the end-point of the powered flight arc at t_B and we call this the "thrust induced drag." It simply manifests like a pressure force that acts from behalf of the surrounding air on the exit area of the rocket motors, toward the direction of motion.

The exit area is usually of the same order of magnitude with the main cross area of the vehicle and the result is a non negligible counter-pressure force that quickly reduces the drag at engine cut-off by 5÷20 percent. It is a known but hard to manage fact. The drag cut-off reduction[14] precisely equals the thrust induced drag,[16]

$$\Delta R \equiv \int_{\varsigma_e} n \cdot \tau \, d\varsigma \approx \int_{\varsigma_e} \rho n \, d\varsigma, \tag{22}$$

with the static pressure p or stress tensor $\boldsymbol{\tau}$ of the perturbed air in the wake. From a variational point of view, this jump of values is disastrous. The unpowered (coast) drag proves significantly different from the powered drag of the same rocket vehicle and this occurrence ends in a discontinuity of order zero-one in motion equations. The coast motion

$$m\frac{d\mathbf{v}}{dt} - m\mathbf{g} - \mathbf{R}_c = 0 \tag{23}$$

is not deducible from the powered flight by simply putting $dm/dt = 0$,

$$m\frac{d\mathbf{v}}{dt} - c\frac{dm}{dt} - m\mathbf{g} - \mathbf{R}_b = 0, \tag{24}$$

because we saw that essentially

$$\mathbf{R}_b \ne \mathbf{R}_c$$

by up to 20 percent or more. An unavoidable interruption in the variational calculus is produced because the transversality condition, marvelously used by Tsien to render the injection conditions to the coast flight, cannot be actually fulfilled,

$$\mathbf{R}_b(\sigma_B,\dot{\sigma}_B)+\left(\frac{\partial\psi}{\partial\sigma}\right)_B\dot{\sigma}_B+g\neq$$
$$\mathbf{R}_c(\sigma_B,\dot{\sigma}_C)+\left(\frac{\partial\psi}{\partial\sigma}\right)_C\dot{\sigma}_C+g. \tag{25}$$

If the position of the powered arc tip B on the coast trajectory is secured, the transversality condition (25) is never satisfied because the two drags, burn and coast, can never be equal (Figure 18–10).

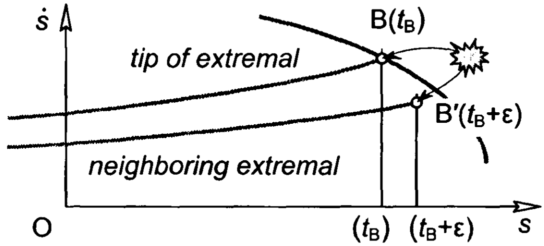

Figure 18–10: Essential of cut-off discontinuity.

If the two drag values are set equal and the condition (25) satisfied, the neighboring extremal arc could never end on the coasting trajectory and the injection connection is impossible. This is a consequence of the fact that the integrand in (17) does not possess continuous derivatives at the right, cut-off end. Such discontinuities are not allowed, and completely new theories are required.

Solution after 85 Years

The solution is called the multiple deviators method and represents a sharp extension of the calculus of variations of Lagrange. The derivatives of the integrand function f do not exist at the end-point because of the jump in the values of f at x_f, when the controls are blocked or stopped, $u(x)=0\big|\, x\geq x_f$,

$$f^{(k)}[x_f-,y(x_f-),\dot{y}(x_f-)]\neq$$
$$\neq f^{(k)}[x_f+,z(x_f+),\dot{z}(x_f+)]. \tag{26}$$

We deal with the fact that the splitting point x_f is moving at the tail of the trajectory, where derivatives simply do not exist and the integrand f ceases to exist after point x_f.

To fix the ideas we remain within the one-dimensional case ($n = 1$). When for $k = 0$ the discontinuity has also the special form

$$f[x_f-, y(x_f-), \dot{y}(x_f-)] > f[x_f+, z(x_f+), \dot{z}(x_f+)]\text{,} \tag{27}$$

And it was shown[21] that the extremal must be shortened to a convenient point B (Figure 18–11). Means must be provided to find the optimal position of B. The solution for rocket vehicles is typical and best serves the answer. This way a new, completely independent variation of a local kind is produced along the extremal arch $\sigma(x)$.

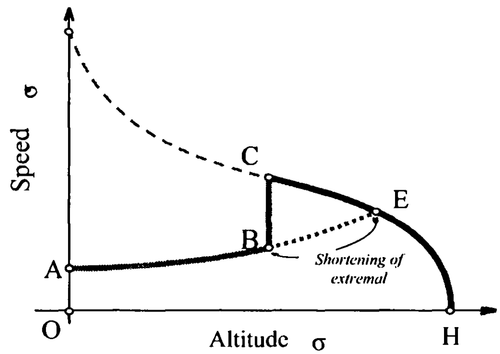

Figure 18–11: Impulsive extremal.

The aspect and scale of the extremal arc are yet to be determined. Under the action of the local deviator ε_h put into action (Figure 18–12) a non-conventional, doubly deviated quasi-extremal $s(x)$ is obtained. The effects of its variation on the extremal are to be subjected to equally nonconventional optimality conditions.

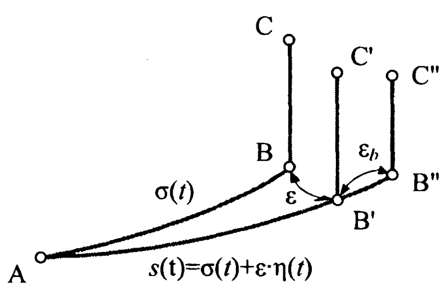

Figure 18–12: The action of two deviator solvers.

The first Lagrange type deviator ε applies to the whole extremal arc $E(x)$,

$$s(x) = \sigma(x) + \varepsilon \cdot \eta(x), \tag{28}$$

while the new, specific deviator ε_b is used to modify the length only of the extremal. It manifests a decidedly local effect and thus a very nonconventional, combined variational problem appears, of global-local type. While it contradicts in some way the global character of the variational method of Lagrange, it offers yet a very new and unexpected extension of this calculus to the problems with discontinuous integrands.

At the same time the deviator function $\eta(x)$ is still considered as arbitrary, except for the fact that it fulfills an identical continuity restriction as the extremal itself. The variable σ is the one-dimensional correspondent of y and the set $\{\varepsilon, \varepsilon_b\}$ contains the multiple-deviators that solve the discontinuous variational problem. The twin deviators offer a sharp extension of Lagrange's method to problems completely inaccessible yet to the classical theory.

Single-Staged Vehicle

The demo application of the new technique is to the optimal control of atmospheric ascent of reactive vehicles. As a first example, the single stage rocket transporter is analyzed and the extremal that minimizes the total lift-off mass of the vehicle that climbs into the atmosphere on a rectilinear path, searched. The equation of motion for the powered flight can be written, for example, in the Hamel form

$$f + \frac{d(\mu F)}{dx} = 0, \tag{29}$$

where are denoted the functions

$$F(x, \dot{y}) = \exp\frac{\dot{y} + g x \sin\theta}{K}, \quad f(x, y, \dot{y}) = \frac{R_p(y, \dot{y})}{K} F(x, \dot{y}). \tag{30}$$

Here x is the time from lift-off, μ the variable mass ratio of the vehicle, θ the constant climbing angle, g the gravity constant and R_p the retardation for powered flight. The total lift-off mass is given by the following functional, with two deviators, specific to the problem where the free point is the right-end node E (Figure 18–11).

$$J(\varepsilon, \varepsilon_B) = \int_0^{x_B + \varepsilon + \varepsilon_B} f[x, y + k(\varepsilon)\eta(x), \dot{y} + \\ + k(\varepsilon)\dot{\eta}(x)]dx + \mu_B F(x_B + \varepsilon_B, \dot{y}_B). \tag{31}$$

The shortened control subarc AB ends in the node B which, in accordance with the dual variation $\{\eta, \varepsilon, \varepsilon_B\}$, manifests the following variations for its position s_B and speed v_B,

$$s_{B'} \equiv s(x_B + \varepsilon + \varepsilon_B) = \sigma(x_B) + k'(0)\varepsilon\eta(x_B) + \\ + \dot{\sigma}(x_B)(\varepsilon + \varepsilon_B) + \frac{\ddot{\sigma}(x_{B-})}{2}(\varepsilon + \varepsilon_B)^2 + \ldots \tag{32}$$

$$v_{B'} \equiv v(x_B + \varepsilon + \varepsilon_B) = \dot{\sigma}(x_B) + k'(0)\varepsilon\dot{\eta}(x_B) + \\ + \ddot{\sigma}(x_{B-})(\varepsilon + \varepsilon_B) + \frac{\dddot{\sigma}(x_{B-})}{2}(\varepsilon + \varepsilon_B)^2 + \ldots \tag{33}$$

Note that the variation of the node C, where the controlled motion ends after the impulsive subarc BC is consumed, is closely related to the variation of motion on the controlled subarc AB. The Hamel form of this equation can be developed in the more explicit form as

$$\frac{dv_B}{dx_B} + K\frac{d\ln m_B}{dx_B} + r = 0, \tag{34}$$

with m_B the local mass, K its vacuum specific thrust and r the resultant of the retarding counter-forces per unit mass. Along the impulsive subarc BC no retardation is felt and the following equation holds (Figure 18–12.)

$$\frac{d(v_C - v_B)}{ds_B} - K\frac{d\ln m_B}{ds_B} = 0. \tag{35}$$

From the relations (34) and (35) the following remarkable equation results, which links the speed at C to the motion parameters along E

$$\frac{dv_C}{ds_B} = -\frac{r_B}{v_B}. \tag{36}$$

Hence the variation in position and in speed at insertion point C writes

$$s_{C'} = s_{B'}, \tag{37}$$

$$v_{C'} \equiv v_C(x_B + \varepsilon + \varepsilon_B) = \dot{\sigma}_C(x_B) + \\ + k'(0)\varepsilon\dot{\eta}(x_B) + \ddot{\sigma}(x_{B-})\varepsilon - r_B\varepsilon_B. \tag{38}$$

The end of powered motion in C must fit the specific zero-control trajectory (S) that, on its turn, provides the desired ceiling. Considering (32) and (33), the first variation of position in node C becomes

$$s_{C'} - \sigma(x_C) \cong k'(0)\varepsilon\eta(x_B) + \dot{\sigma}(x_B)(\varepsilon + \varepsilon_B), \tag{39}$$

while the corresponding variation in speed is

$$v_{C'} - \dot{\sigma}_C(x_B) = k'(0)\varepsilon\dot{\eta}(x_B) + \ddot{\sigma}(x_{B-})\varepsilon - r_B\varepsilon_B. \tag{40}$$

Meanwhile, there also exists an allowed variation of the insertion point along S,

$$v_{C'} = \dot{\sigma}_C (x_B) + \left(\frac{\partial \psi}{\partial \sigma}\right)_C (s_{C'} - \sigma_C) + O(s_{C'} - \sigma_C)^2 \tag{41}$$

These variations must coincide. Substituting equations (39) and (40) in (41), rounded to the first order approximation, one obtains the rough form of the variational condition for insertion on the coast subarc S

$$\begin{aligned} &k'(0)\,\varepsilon\,\dot{\eta}(x_B) + \ddot{\sigma}(x_{B-})\,\varepsilon - r_B \varepsilon_B = \\ &= \left(\frac{\partial \psi}{\partial \sigma}\right)_C \{[k'(0)\,\eta(x_B) + \dot{\sigma}(x_B)]\,\varepsilon + \dot{\sigma}(x_B)\,\varepsilon_B\} \end{aligned} \tag{42}$$

This insertion condition must be satisfied for any arbitrary and independent values of the two deviators ε and ε_B. This can only happen when the coefficients of both deviators vanish simultaneously. Two equations appear

$$\left(\frac{\partial \psi}{\partial \sigma}\right)_C \dot{\sigma}(x_B) + r_B = 0\,, \tag{43}$$

$$\begin{aligned} &\left(\frac{\partial \psi}{\partial \sigma}\right)_C [k'(0)\,\eta(x_B) + \dot{\sigma}(x_B)] - \\ &- k'(0)\,\dot{\eta}(x_B) - \ddot{\sigma}(x_{B-}) = 0\,. \end{aligned} \tag{44}$$

The first is eventually exploited, but note now that, at the same time, another condition for the linear coefficient $k'(0)$ appears when the first variation of the functional (31) in respect to the global deviator ε is set to zero

$$\begin{aligned} \left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=\varepsilon_B=0} &\equiv \int_0^{x_C} \eta(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] dx + \\ &+ k'(0)\,\eta(x_B)\left(\frac{\partial f}{\partial \dot{y}}\right)_B + + f(x_B, y_B, \dot{y}_B) + \\ &+ \left(\frac{\partial F}{\partial \dot{y}}\right)_C [\ddot{\sigma}(x_{B-}) + k'(0)\,\dot{\eta}(x_B)] + \left(\frac{\partial F}{\partial x}\right)_C = 0. \end{aligned} \tag{45}$$

Besides the Euler equation (square brackets factor in the integrand) that remains identical with the continuous form, the transversality condition gets a very specific aspect

$$k'(0)\left[\eta(x_B)\left(\frac{\partial f}{\partial \dot{y}}\right)_B + \dot{\eta}(x_B)\left(\frac{\partial F}{\partial \dot{y}}\right)_C\right] + \\ + f(x_B) + \left(\frac{\partial F}{\partial \dot{y}}\right)_C \ddot{\sigma}(x_{B-}) + \left(\frac{\partial F}{\partial x}\right)_C = 0. \tag{46}$$

The unknown constant $k'(0)$ is eliminated from equations (44) and (46), resulting in

$$\left[\ddot{\sigma}(x_{B-}) - \left(\frac{\partial \psi}{\partial y}\right)_C \dot{\sigma}(x_B)\right]\left[\eta(x_B)\left(\frac{\partial f}{\partial \dot{y}}\right)_B + \\ + \dot{\eta}(x_B)\left(\frac{\partial F}{\partial \dot{y}}\right)_C\right] + \left[\left(\frac{\partial \psi}{\partial y}\right)_C \eta(x_B) - \dot{\eta}(x_B)\right] \\ \left[f(x_B) + \left(\frac{\partial F}{\partial \dot{y}}\right)_C \ddot{\sigma}(x_{B-}) + \left(\frac{\partial F}{\partial x}\right)_C\right] = 0. \tag{47}$$

This condition must be satisfied for arbitrary function $\eta(x)$, only possible when the coefficients of $\eta(x)$ and $\dot{\eta}(x)$ are vanishing simultaneously. This ends in two equations

$$\ddot{\sigma}(x_{B-})\left(\frac{\partial f}{\partial \dot{y}}\right)_B + \left[f(x_B) + \ddot{\sigma}(x_{B-})\left(\frac{\partial F}{\partial \dot{y}}\right)_C + \\ + \left(\frac{\partial F}{\partial x}\right)_C - \dot{\sigma}(x_B)\left(\frac{\partial f}{\partial \dot{y}}\right)_B\right]\left(\frac{\partial \psi}{\partial y}\right)_C = 0, \tag{48}$$

$$\dot{\sigma}(x_B)\left(\frac{\partial \psi}{\partial y}\right)_C\left(\frac{\partial F}{\partial \dot{y}}\right)_C + \left(\frac{\partial F}{\partial x}\right)_C + f(x_B) = 0. \tag{49}$$

It may be noticed that the condition (49) is identical to the one already deduced by another route in (43) and can be further developed. The derivative of speed is directly connected to the position on the coast trajectory,

$$\left(\frac{\partial \psi}{\partial y}\right)_C \equiv \left(\frac{\partial \dot{y}}{\partial y}\right)_C = -\frac{r_C}{\dot{y}_C}\dot{y}_C \tag{50}$$

and thus the remarkably simple and specific transversality condition holds for the extreme points of the discontinuous functional

$$\dot{y}_B r_C = \dot{y}_C r_B\,. \tag{51}$$

or in its extended form

$$-\frac{D_C + g}{v_C} = \frac{\frac{R_B}{\mu_B} + g}{v_B}. \tag{52}$$

This equation gives the amplitude of the final impulse of the extremal and the very moment when this subarc intervenes, supplying the complete solution of the discontinuous integrand problem.

Conclusion

The problem initially formulated by Goddard could not have been solved for almost 85 years with the means of the classical calculus of variations. The reason resides in the fact that the classical theory in the calculus of variations can only cover the continuous problems, while that of the atmospheric rocket ascent is specifically discontinuous. All former solutions of optimal flight control were consequently partial only, being structured by ignoring the cut-off jump in derivatives for the flight of rocket vehicles. It was supposed that the errors introduced this way are small enough that the partial solutions are acceptable.

After 85 years, the means were found, at last, to extend the variational calculus by the so-called multiple deviators method, up to the point where the Goddard problem is correctly and completely solved.

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