TECHNICAL TRANSLATION
F-44

THE ATTAINABILITY OF HEAVENLY BODIES

By Walter Hohmann

Translation of "Die Erreichbarkeit der Himmelskörper."
R. Oldenbourg (Munich - Berlin), 1925.

CONTENTS

The prese.it work will contribute to the recognition .that space travel is to be taken seriously and that the final successful solution of the problem cannot be doubted, if existing technical possibilities are purposefully perfected as shown by conservative mathematical treatment.

In the original work 10 years ago, the author believed that 2,000 m/sec was the highest gas velocity attainable for the foreseeable future by our technical means. Calculations originally were only carried out for this value. But in the meantime three works concerning the rocket problem were published, which make it apparent that far higher exhaust-gas velocity can be reached with a suitable array:

Robert H. Goddard: "A Method of Reaching Extreme Altitudes" (mainly on the basis of practical experiments);

Herman Oberth: "The Rocket Into Interspace" (especially valuable for detailed suggestions on the basis of theory);

Max Valier: "The Advance Into Space" (a simple presentation of the problem).

Therefore and for direct comparison with Oberth's results, calculations were extended to higher gas velocity (2,500, 3,000, 4,000, and 5,000 m/sec), leaving the original 2,000 m/sec now as the lower limit. This makes conditions much more favorable; however, the following observations apply:

The use of low gas velocities required avoidance of any deadweight. This led to arranging the fuel to be carried with the rocket in the shape of a cylinder of solid fuel, whose combustion would automatically result in the escape of the gas at the required velocity. This arrangement represents the ideal solution, because no deadweight is involved, but is conceivable only at low gas velocities. According to Oberth the higher velocity can be achieved only by means of gases escaping through narrowed jets during combustion; carrying the jets and the container for the liquid fuel means a more or less large deadweight, which is easier to carry at higher gas velocity.

The weights used in the last two sections do not include these unavoidable deadweights, because estimation without practical experiments with jets and containers is hardly possible. Quoted weights to be lifted G_0 therefore represent the lowest value while using an ideal fuel.

Taking higher velocities into consideration as well as later supplements - particularly investigations concerning possibility of landing without deceleration ellipse at the end of section II, and concerning

intersecting ellipses at the end of section V, also considerations concerning heating during landing - are due to suggestions by Mr. Valier and Professor Oberth.

Since the writer is an engineer, not a mathematician, clumsy approximations in place of mathematical formulas occasionally appear in the calculations; this should not affect the results.

Essen, October 1925   W. Hohmann

THE ATTAINABILITY OF HEAVENLY BODIES[*]

By Walter Hohmann

I

LEAVING THE EARTH

In free space we could impart to our vehicle an arbitrary velocity Δv by expelling a part Δm of the vehicle's mass m with the velocity c in opposite direction relative to the vehicle. Since the center of gravity (CG) of the total mass m remains at rest, we have after time t according to Figure 1:
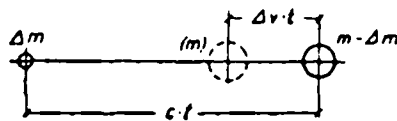

Figure 1

$$\Delta m(c \cdot t - \Delta v \cdot t) = (m - \Delta m) \cdot \Delta v \cdot t;$$

or

$$\frac{m - \Delta m}{\Delta m} = \frac{c - \Delta v}{\Delta v} ,$$

or

$$\frac{m}{\Delta m} = \frac{c}{\Delta v} , \qquad\qquad (1)$$

thus

$$\Delta v = c \cdot \frac{\Delta m}{m} ;$$

i.e., once Δm moves with velocity c, the remaining mass $(m - \Delta m)$ moves with velocity $\Delta v = c \cdot \frac{\Delta m}{m}$ opposite to Δm, as long as nothing else happens.

If mass is expelled or "radiated" at the rate $\frac{dm}{dt}$, then the remaining mass has acceleration:

$$\frac{dv}{dt} = \frac{c}{m} \cdot \frac{dm}{dt} \qquad (1a)$$

while the mass decreases steadily.

If we arrange operations such that the radiated mass $\frac{dm}{dt}$ is always proportional to the mass still present, so that

$$\frac{dm}{dt} : m = \alpha = \text{constant}$$

then the acceleration is uniform and independent of the mass:

$$\frac{dv}{dt} = c \cdot \alpha, \qquad (1b)$$

providing c is unchanged.

The mass decreases according to

$$\frac{dm}{dt} = -\alpha m \qquad (1c)$$

(negative, since m decreases with time), thus

$$\int \frac{dm}{m} = -\alpha \int dt$$

and after integration

$$\ln m = -\alpha t + C.$$

If m_0 is the mass at $t = 0$, then

$$\ln m_0 = 0 + C;$$
$$C = \ln m_0;$$

thus

$$\ln m = -\alpha t + \ln m_0,$$

or

$$\ln \frac{m}{m_0} = -\alpha t,$$

and

$$\frac{m}{m_0} = e^{-\alpha t} \quad \text{or} \quad \frac{m_0}{m} = e^{\alpha t};$$ (2)

i.e., after time t there remains

$$m = \frac{m_0}{e^{\alpha t}}.$$

If now a gravitational acceleration g opposes the acceleration α of a vehicle such as described above, then the resulting acceleration is

$$\frac{dv}{dt} = \alpha - g.$$

If f.i. the vehicle moves radially outwards at a distance r from the center of the Earth and if g_0 is the earth's acceleration, corresponding to r_0 (Figure 2), then in accordance with the law of gravity (derived at the end of section III):
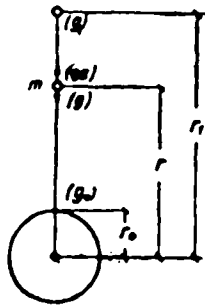

Figure 2

$$g = g_0 \cdot \frac{r_0^2}{r^2}$$ (3)

thus the total acceleration is:

$$\frac{dv}{dt} = c\alpha - g_0 \frac{r_0^2}{r^2};$$

and since

$$\frac{dr}{dt} = v$$

there follows

$$\frac{dv}{dr} = \frac{\alpha - \frac{g_0 r_0^2}{r^2}}{v} \; ; \quad \int v\,dv = \int (\alpha - \frac{g_0 r_0^2}{r^2})dr; \quad \frac{v^2}{2} = \alpha r + \frac{g_0 r_0^2}{r} + C.$$

If acceleration is to begin from rest ($v = 0$) at $r = r_0$, then

$$0 = \alpha r_0 + \frac{g_0 r_0^2}{r_0} + C,$$

thus

$$C = -\alpha r_0 - g_0 r_0 = -r_0(\alpha + g_0),$$

therefore in general

$$\frac{v^2}{2} = \alpha r + \frac{g_0 r_0^2}{r} - r_0(\alpha + g_0) = (r - r_0)(\alpha - g_0 \frac{r_0}{r}) \qquad (4)$$

If at r_1, after reaching maximum velocity v_1, acceleration α ceases, then the vehicle will subsequently behave like a body thrown upward with velocity v_1, i.e., its instantaneous velocity at $r' > r_1$

$$v' = \frac{dr'}{dt}$$

diminishes at the rate

$$\frac{dv'}{dt} = -g_0 \frac{r_0^2}{r'^2} \; ;$$

these two equations give

$$v'\,dv' = -g_0 r_0^2 \frac{dr'}{r'^2} \; ;$$

thus

$$\frac{v'^2}{2} = + \frac{g_0 r_0^2}{r'} + C;$$

where

$$C = \frac{v_1^2}{2} - \frac{g_o r_o^2}{r_1} ,$$

or

$$\frac{v'^2}{2} = \frac{g_o r_o^2}{r'} + \frac{v_1^2}{2} - \frac{g_o r_o^2}{r_1} . \qquad (5)$$

If the vehicle is to reach its highest velocity v_1 at r_1 such that it does not return because of gravity after acceleration $c\alpha$ ceases, then the final velocity $v' = 0$ must be reached only at $r' = \infty$, so that according to equation (5)

$$\frac{v_1^2}{2} = \frac{g_o r_o^2}{r_1} ; \qquad (6)$$

on the other hand by equation (4)

$$\frac{v_1^2}{2} = c\alpha r_1 + \frac{g_o r_o^2}{r_1} - r_o(c\alpha + g_o);$$

therefore

$$c\alpha r_1 = r_o(c\alpha + g_o),$$

or

$$r_1 = r_o\frac{c\alpha + g_o}{c\alpha} = r_o(1 + \frac{g_o}{c\alpha}) \qquad (7)$$

and

$$v_1 = \sqrt{\frac{2g_o r_o^2}{r_1}} = \sqrt{\frac{2g_o r_o}{1 + \frac{g_o}{c\alpha}}} . \qquad (8)$$

The duration t_1, after which r_1 and maximum velocity v_1 are attained, follows from

$$\frac{dr}{dt} = v$$

in the general form

$$t_1 = \int_{r_o}^{r_1} \frac{dr}{v} = \int_{r_o}^{r_1} \frac{dr}{\sqrt{2c\alpha r + \frac{2g_o r_o^2}{r} - 2r_o(c\alpha + g_o)}} .$$

Since the solution of this integral presents difficulties, the variation of the gravitational acceleration g during the time t_1 will be neglected and g will be replaced by a mass value g_m between g_0 and g_1 and, to be conservative, g_m will be taken not as $\frac{g_0 + g_1}{2}$, but

$$g_m = \frac{2g_0 + g_1}{3},$$

or taking into account equation (3)

$$g_m = \frac{2g_0 + g_0 \frac{r_0^2}{r_1^2}}{3} = \frac{g_0}{3}(2 + \frac{r_0^2}{r_1^2})$$

(for small values of αc, this mean is too favorable, more correct would be the general form

$$g_m = \frac{\xi \cdot g_0 + g_1}{\xi + 1},$$

where f.i. ξ might be taken $\xi = \frac{r_0}{2r_0 - r_1}$, so that for $\alpha c = g_0$ the entire acceleration β would actually $= 0$.) i.e., the duration is to be calculated for an acceleration

$$\beta = c\alpha - \frac{g_0}{3}(2 + \frac{r_0^2}{r_1^2}) \tag{9}$$

in place of $c\alpha - g_0 \frac{r_0^2}{r^2}$. We have then approximately, making use of (7) and (8),

$$t_1 = \frac{v_1}{\beta} = \frac{v_1}{c\alpha - \frac{g_0}{3}(2 + \frac{r_0^2}{r_1^2})} = \frac{\sqrt{\frac{2g_0 r_0}{1 + \frac{g_0}{c\alpha}}}}{c\alpha - \frac{g_0}{3}(2 + \frac{1}{(1 + \frac{g_0}{c})^2})} . \tag{10}$$

If the value t_1, thus formed, is used in (2), there follows

$$\frac{m_1}{m_0} = e^{-\alpha t_1} \quad \text{or} \quad \frac{m_0}{m_1} = e^{\alpha t_1} \tag{11}$$

giving the relation between the mass m_0 at the beginning of the acceleration period t_1 and the mass m_1, remaining at the end of that duration. The difference $m_0 - m_1$ is so to say dead weight and has to be expelled with uniform velocity c during this time to impart to m_1 the velocity v_1, which it must have at r_1.

TABLE I

Acceleration cα . . . . (m/sec²)	15	20	25	30	40	50	100	200
$r_1 = r_0\left(1 + \frac{g_0}{c\alpha}\right)$ . . . . (km)	10,600	9,510	8,860	8,490	7,950	7,640	7,000	6,680
$v_1 = \sqrt{\dfrac{2g_0 r_0}{1 + \frac{g_0}{c\alpha}}}$ . . . . (m/sec)	8,660	9,150	9,470	9,680	10,000	10,200	10,650	10,890
$\beta = c\alpha - \dfrac{g_0}{3}\left(2 + \dfrac{r_0^2}{r_1^2}\right)$ . . . . . . (m/sec²)	7.27	12.00	16.76	21.61	32.35	41.18	90.76	190.46
$t_1 = \dfrac{v_1}{\beta}$ . . . . (sec)	1,192	762	565	448	319	248	117	57
Ratio $\frac{m_0}{m_1} = e^{\alpha t_1}$ for velocity c c = 1,000 m/sec	58,700,000	4,160,000	1,545,000	675,000	346,000	240,000	120,300	89,130
c = 1,500 m/sec	149,000	25,000	12,000	7,750	4,950	3,840	2,400	2,000
c = 2,000 m/sec	7,570	2,010	1,160	825	587	495	547	299
c = 2,500 m/sec	1,270	438	282	216	164	143	108	95.5
c = 3,000 m/sec	388	159	110	88	70	62	49	44.7
c = 4,000 m/sec	87.3	44.8	34.1	28.7	24.2	22.2	18.7	17.2
c = 5,000 m/sec	35.7	20.9	16.7	14.6	12.8	11.9	10.4	9.8
c = 10,000 m/sec	6.0	4.6	4.1	3.8	3.6	3.5	3.2	3.1

Thus m_1 constitutes the payload liberated from the Earth's gravitation. If it is chosen according to practical considerations together with c and acceleration cα -- and thus also the value α, then r_1, v_1, t_1 and m_0 follow from equations (7), (8), (10) and (11).

Table I permits a survey of the influence of values c and cα on the relation of m_0 to m_1. The assumptions were

$$r_o = 6,380 \text{ km and } g_o = 9.8 \text{ m/sec}^2 = 0.0098 \text{ km/sec}^2$$

(the results represent only approximations).

The table shows, that the effect of $c\alpha$ is relatively smaller than that of c, therefore the highest possible jet velocity c is of prime importance, while the acceleration $c\alpha$ may be chosen as high as is possible to withstand. This acceleration is felt by the occupants as an increase in gravity and is thus limited by physiological considerations. The following consideration will serve to find a suitable value. A man jumping from a height of h = 2 m has a velocity $v = \sqrt{2hg_o}$, when touching the ground. From that moment he decelerates this velocity to zero within a distance of h' = 0.5 m and consequently will feel the acceleration β, given by $v = \sqrt{2h'\beta}$, as additional gravity. These two equations give

$$\beta = g_o \frac{h}{h'} = g_o \frac{2.0}{0.5} = 4g_o = \sim 40 \text{ m/sec}^2.$$

If one takes into consideration, that during space travel the acceleration will act over a span of several minutes, while in our example its duration is limited to a fraction of a second, then an acceleration $c\alpha$ of about 20 to 30 m/sec^2 would seem just feasible (More detailed investigations concerning the physiological effect of acceleration are carried out in "The Rocket Into Space" by Oberth).

More difficult to meet is the requirement of the highest possible velocity c. The highest velocity now available to man is that of an artillery shell, say 1,000 to 1,500 m/sec. But because of the high value of m_o/m_1, this cannot be considered here, as may be seen from Table I; rather we have to demand at least a value c of about 2,000 m/sec.

Thus the ratio $\frac{m_o}{m_1} = 825$ with c = 2,000 m/sec and $c\alpha = 30$ m/sec^2 is the least required by these considerations.

This lower limit ($c\alpha = 30$; c = 2,000) is used for the following calculations, with higher values of c being occasionally used for comparison.

The mass to be expelled/sec at the beginning of departure is in accordance with (1c)

$$\frac{dm_o}{dt} = \alpha \cdot m_o ,$$

where

$$\alpha = \frac{\alpha}{c} = \frac{30 \text{ m/sec}^2}{2{,}000 \text{ m/sec}} = \frac{0.015}{\text{sec}} \ ;$$

and

$$m_0 = 825 \ m_1;$$

so that

$$\frac{dm_0}{dt} = 0.015 \cdot 825 \ m_1 = 12.4 \ m_1.$$

At the beginning, considerable amounts of mass have to be radiated, compared to the payload. If one were to expel this mass by firing artillery, one would have to carry correspondingly heavy pieces, thus in turn increasing the payload m_1 and again in turn the total m_0. To avoid this, the operational mass $m_0 - m_1$ may be arranged so to say like a rocket, that is burning slowly, while the combustion products are expelled with the required velocity c into a space, assumed empty. Since the combustion mass rate is now proportional to the rocket cross section and since -- by (1c) -- it also must be proportional to the mass remaining above the instantaneous cross section, we must imagine the cross section to be proportional to the mass resting above it, so that the fuel would be arranged in the form of a tower of constant weight/area ratio (see Figure 3).
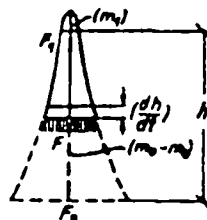

Figure 3

The mass to be expelled/sec at F is, according to (1c) and Figure 3:

$$\frac{dm}{dt} = \alpha m = F \cdot \frac{dh}{dt} \cdot \frac{\gamma'}{g_0} \ ,$$

where g_0 and γ' denote density of the fuel and gravitational acceleration at the earth surface respectively. Therefore

$$\frac{dh}{dt} = \frac{am}{F} \cdot \frac{g_o}{\gamma'} \, ,$$

or, since

$$\frac{m}{F} = \frac{m_1}{F_1} = \frac{m_o}{F_o} : \qquad (12)$$

$$dh = \frac{am_1}{F_1} \cdot \frac{g_o}{\gamma'} \cdot dt,$$

and

$$h = \frac{am_1}{F_1} \cdot \frac{g_o}{\gamma'} \cdot \int_o^{t_1} dt = \frac{am_1}{F_1} \cdot \frac{g_o}{\gamma'} \cdot t_1,$$

or if $G_1 = m_1 \cdot g_o$ denotes the weight at the Earth surface of the payload m_1, then:

$$h = \frac{at_1}{\gamma'} \cdot \frac{G_1}{F_1} \, . \qquad (12a)$$

Further, by (12):

$$F_o = \frac{m_o}{m_1} \cdot F_1.$$

If f.i. the weight to be lifted $G_1 = 2t$ and of the fuel density $\gamma' = 1.5t/m^3$, then for the assumed case ($c\alpha = 30 \text{ m/sec}^2$; $c = 2,000$ m/sec; $\alpha = \dfrac{0.015}{\text{sec}}$; $t_1 = 448$ sec; $\dfrac{m_o}{m_1} = 825$) there follows the relation:

$$h = \frac{0.015 \cdot 448}{1.5} \cdot \frac{2.0}{F_1} = \frac{8.96}{F_1} \, ;$$

$$F_o = 825 \cdot F_1;$$

and if one assumes an upper cross section for the tower of $F_1 = 0.332 \text{ m}^2$ corresponding to a 0.65 m dia.:

$$F_o = 825 \cdot 0.332 = 273 \text{ m}^2, \text{ corresponding to } 18.7 \text{ m} \phi,$$
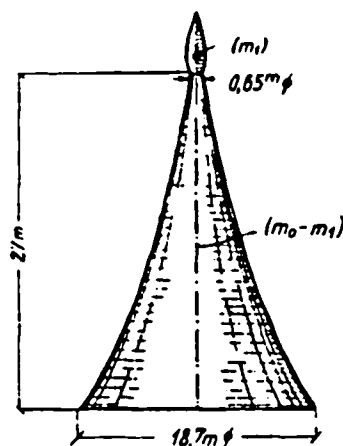
$$h = \frac{8.96}{0.332} = 27 \text{ m (see Figure 4)}.$$

Figure 4

The force on the material, considering $c\alpha = 30$ m/sec^2 instead of the usual $g_0 = 9.8$ m/sec^2, is:

$$\sigma = \frac{c\alpha}{g_0} \cdot \frac{G_1}{F_1} = \frac{30}{9.8} \cdot \frac{2t}{0.332 \text{ m}^2} = 18.5 \text{ t/m}^2 = 1.85 \text{ kg/cm}^2.$$

To produce a material, having the necessary strength as well as the necessary energy of combustion to produce velocity c, is a question for the technology of explosives.

So far, air resistance has been ignored. Even if the shape of the vehicle (Figure 4) is favorable for overcoming atmospheric friction and even if higher velocities are achieved only at heights, where very little or no atmosphere is present, the influence of the lower and denser regions must at least approximately be taken into consideration.

According to von Loessl, the resistance W of air of specific density γ with respect to a body, moving with velocity v at right angles to its cross section F, is:

$$W = \frac{\gamma v^2}{g} \cdot F \cdot \psi \text{ (eq. (14), section II),}$$

where g is the Earth acceleration and ψ has a value depending on the body's geometry (for vertically effected areas $\psi = 1$). The resulting retardation is

$$\Delta\beta = \frac{W}{m} = \frac{\gamma v^2}{g} \cdot \frac{F}{m} \cdot \psi.$$

In the present case according to (12)

$$\frac{F}{m} = \text{constant} = \frac{F_1}{m_1} = \frac{0.332}{2,000/10} = \frac{1}{600} \frac{m^3}{\text{kg/sec}^2} \; ;$$

further, according to Figure 5, we may approximate by a cone

$$\psi = \sin^2 \varphi = \sim (\frac{18.7}{2 \cdot 27})^2 = 0.12,$$

so that

$$\Delta\beta = \frac{\gamma v^2}{g} \cdot \frac{0.12}{600} = \frac{\gamma v^2}{g} \cdot \frac{1}{5,000} \, . \tag{13}$$
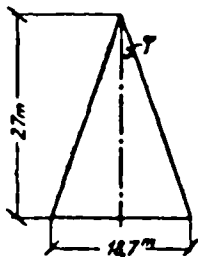

Figure 5

In the range considered, we have with sufficient accuracy

$$g = \sim 10 \text{ m/sec}^2$$

and according to (4):

$$v^2 = 2(r - r_0)(c\alpha - g_0 \frac{r_0}{r}).$$

The values for γ may be found in Table III (page 19), section II, and accordingly in Table II the values for $\frac{\gamma v^2}{g}$ in kg/m^2 are calculated for various values of the distance r.

Table II

r km	$(r - r_o)$ km	$(c\alpha - g_o \frac{r_o}{r})$ km/sec^2	v^2 km^2/sec^2	γ (from Table III) kg/m^3	$\frac{\gamma v^2}{g}$ kg/m^2
6,380	0	0.02020	0.00	1.30	0
6,381	1	0.02020	0.04	1.15	4,600
6,382	2	0.02020	0.08	1.00	8,000
6,383	3	0.02020	0.122	0.90	11,000
6,384	4	0.02020	0.162	0.80	13,000
6,385	5	0.02020	0.202	0.70	14,200
6,386	6	0.02020	0.243	0.62	15,100
6,388	8	0.02021	0.323	0.48	15,500
6,390	10	0.02021	0.404	0.375	15,200
6,395	15	0.02022	0.606	0.215	13,000
6,400	20	0.02023	0.810	0.105	8,500
6,410	30	0.02024	1.214	0.0283	3,440
6,420	40	0.02026	1.620	0.0074	1,200
6,430	50	0.02027	2.028	0.00187	370
6,440	60	0.02028	2.434	0.00045	110
6,460	80	0.02032	3.250	0.000023	7.5
6,480	100	0.02035	4.070	0.000001	0.4

Above 50 km from the Earth's surface the air resistance according to (13) is no longer significant at the velocities attained there. To take an unfavorable case, we will assume an average value of

$$\frac{\gamma v^2}{g} = 12,000 \text{ kg/m}^2$$

14

for the region between 0 and 50 km, so that the mean retardation from (13) is

$$\Delta\beta = \frac{12,000}{5,000} = 2.4 \text{ m/sec}^2$$

leaving an effective acceleration of only

$$\alpha - \Delta\beta = 30 - 2.4 = 27.6 \text{ m/sec}^2$$

below the first 50 km in place of $c\alpha = 30$ m/sec^2.

In $r = 6,430$ km or $r - r_0 = 50$ km height, we have therefore according to (4):

$$\frac{v^2}{2} = 50(0.0276 - 0.0098 \frac{6,380}{6,430}) = 0.895 \text{ km}^2/\text{sec}^2$$

in place of

$$50(0.03 - 0.0098 \cdot \frac{6,380}{6,430}) = 1.014 \text{ km}^2/\text{sec}^2$$

or

$$v = \sqrt{2 \cdot 0.895} = 1.340 \text{ km/sec}$$

in place of

$$\sqrt{2 \cdot 1.014} = 1.425 \text{ km/sec}$$

and the time up to that point

$$t' = \frac{1,340}{27.6 - \frac{9.8}{3}(2 + \frac{6,380^2}{6,439^2})} = 75 \text{ sec}$$

in place of

$$\frac{1,425}{30 - \frac{9.8}{3}(2 + \frac{6,380^2}{6,430^2})} = 70.3 \text{ sec}$$

the time difference therefore $\Delta t = 4.7$ sec.

Since further the final velocity turns out too low by

$$\Delta v' = 1.425 - 1.340 = 0.085 \text{ km/sec}$$

the acceleration has to act longer for a time

$$\Delta t' = \frac{\Delta v'}{\beta'} = \frac{85}{30 - 9.8 \cdot \frac{6,380^2}{6,490^2}} = 3.5 \text{ sec}$$

resulting in a propulsion time

$$t_1' = 448 + 4.7 + 3.5 = 456 \text{ sec};$$

in place of $t_1 = 448$ sec; therefore

$$\alpha t_1' = 0.015 \cdot 456 = 6.84$$

and the ratio

$$\frac{m_0}{m_1} = e^{\alpha t_1'} = 933 \text{ in place of } 825.$$

The result becomes somewhat more favorable, if the acceleration below the first 50 km is simply increased by $\Delta\beta = 2.4$ m/sec^2, the propulsion period then remaining at 448 sec, during the first 70.3 sec of which $\alpha c = 32.4$ m/sec^2 with $\alpha = \frac{32.4}{2,000} = 0.0162$, while during the remaining 377.7 sec $\alpha c = 30$ m/sec^2, corresponding to an $\alpha = 0.015$, so that

$$\frac{m_0}{m_1} = e^{\Sigma \alpha t} = e^{0.0162 \cdot 70.3 + 0.015 \cdot 377.7} = 898.$$

The table below shows the air resistance calculated similarly but for different αc and c:

m/sec	$\alpha c = 30$ m/sec^2 ($t_1' = 456$ in place of 448 sec)	$\alpha c = 100$ m/sec^2 ($t_1' = 123$ in place of 117 sec)	$\alpha c = 200$ m^{2}sec ($t_1' = 64$ in place of 57 sec)
c = 2,000	933 in place of 825	468 in place of 342	602 in place of 299
c = 2,500	235 in place of 216	138 in place of 108	166 in place of 95.5
c = 3,000	95 in place of 88	60 in place of 49	71 in place of 44.7

m/sec	αc = 30 m/sec^2 (t_1' = 456 in place of 448 sec)	αc = 100 m/sec^2 (t_1' = 123 in place of 117 sec)	αc = 200 m^{2}sec (t_1' = 64 in place of 57 sec)
c = 4,000	30 in place of 28.7	22 in place of 18.7	25 in place of 17.2
c = 5,000	15 in place of 14.6	12 in place of 10.4	13 in place of 9.8

$$\frac{m_0}{m_1} = e^{\frac{\alpha c}{c} t_1'}$$

The effect of the air thus rapidly increases with increasing αc, so that finally high values here can become more unfavorable than lower values of αc, because of the early attainment of high velocities.

The above concept, to impart to a body an acceleration opposing gravity by expelling parts of its mass, is not new in itself. It is contained unwittingly in Jule Verne's "Journey Around the Moon," where mention is made of rockets, which are taken along to decelerate the vehicle, and it is continuously used in Kurd Lasswitz's "On Two Planets." Here however under the very favorable assumption, that the velocity of expulsion is that of light, resulting in only a minor decrease of the vehicle's mass.

The more recent works of Goddard, Oberth and Valier have already been mentioned. Also the pioneer of aviation, Hermann Ganswindt, has proclaimed the idea of a rocket vehicle in public talks around 1890, at the same time also, the Russian Cielkowsky. Finally, Newton mentioned the possibility of travel in empty space on the occasion of a lecture concerning impulse reaction.

II

RETURN TO EARTH

To decelerate a vehicle, as described in the last section (Figure 4), falling toward the center of attraction within the range r_1 and r_0 (Figure 2) from velocity v_1 to a state of rest, we require the same t_1, given by (10), during which now however the particles are expelled in the direction of motion at the rate $\frac{dm}{dt}$.

Thus for a return trip the propulsion period would become twice as great, resulting in a mass ratio $\frac{m'_o}{m_o} = e^{\alpha t_1} \cdot 2$, i.e., the second power of the values quoted in Table I for the ratio $\frac{m_o}{m_1}$, f.i. in place of $c\alpha$ = 30 m/sec^2 and c = 2,000 m/sec:

$$\frac{m_o'}{m_1} = 825^2 = 680,625.$$

This type of retardation -- at least for presently feasible jet velocity c -- would make matters extremely unfavorable. A different type of landing, namely retardation by the Earth atmosphere, must be attempted.

According to von Loessl, the resistance to a body, entering the atmosphere, is

$$W = w \cdot F\psi = \gamma \cdot \frac{v^2}{g} \cdot F\psi, \qquad (14)$$

where v = instantaneous velocity

g = gravity

γ = density of the air

w = pressure/unit area, vertically to the direction of motion

F = area of the body at right angles to the motion

ψ = a constant, depending on the surface geometry, f.i. plane area ψ = 1, convex hemisphere ψ = 0.5.

Setting atmospheric pressure at sea level p_o and zero at height h and assuming (Figure 6) a law

$$p = p_o(\frac{y}{h})^n \qquad (15)$$

then the pressure rises with height dy as

$$\frac{dp}{dy} = \frac{np_o}{h^n} y^{n-1};$$

but

$$dp = \gamma dy \text{ or } \frac{dp}{dy} = \gamma,$$

so that

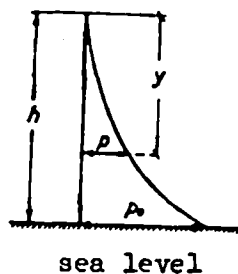$$\gamma = \frac{np_0}{h^n} y^{n-1}. \tag{16}$$

sea level

Figure 6

Since at sea level $y = h$ and $p = p_0$, we have

$$\gamma_0 = \frac{np_0}{h},$$

or

$$n = \frac{\gamma_0}{p_0} \cdot h \tag{17}$$

and according to (16):

$$\gamma = \frac{\gamma_0}{p^0} \cdot h \cdot \frac{p_0}{h^n} \cdot y^{n-1} = \gamma_0 (\frac{y}{h})^{n-1} \tag{16a}$$

Empirically

$\gamma_0 = 1.293 \text{ kg/m}^3$

$p_0 = 0.76 \text{ m} \cdot 13{,}600 \text{ kg/m}^3 = 10{,}330 \text{ kg/m}^2$ (weight of Mercury column)

$$\frac{\gamma_0}{p_0} = \frac{1.293 \text{ kg/m}^3}{10{,}330 \text{ kg/m}^2} = \frac{1}{8{,}000 \text{ m}} = \frac{1}{8 \text{ km}} \tag{17a}$$

Results of balloon experiments show the atmospheric pressure to be 210 mm Mercury at $h - y = 10$ km, or that

$$\frac{p}{p_0} = \frac{210}{760} = \sim \frac{1}{3.6} \, ,$$

which follows from (15) independently of h, providing this height is assumed to lie between 100 and 1,000 km. Meteor observations and theoretical reflections suggest an atmosphere of at least h = 400 km (c.f. Trabert "Lehrbuch der kosmischen Physik," page 304). In the following this value is adopted; from (17) and (17a)

$$n = \frac{400}{8} = 50; \quad n - 1 = 49;$$

and the value of γ corresponding to h - y is given in Table III below.

Table III

h - y km	y km	$\gamma = 1.293\left(\frac{y}{h}\right)^{49}$ kg/m^3	h - y km	y km	$\gamma = 1.293\left(\frac{y}{h}\right)^{49}$ kg/m^3
0	400	1.3	55	345	0.000 915
1	399	1.15	60	340	0.000 448
2	398	1.00	65	335	0.000 217
3	397	0.90	70	330	0.000 102 5
4	396	0.80	75	325	0.000 049 7
5	395	0.70	80	320	0.000 023 0
10	390	0.375	85	315	0.000 010 6
15	385	0.215	90	310	0.000 004 9
20	380	0.105	95	305	0.000 002 2
25	375	0.055	100	300	0.000 000 98
30	370	0.028 3	105	295	0.000 000 423
35	365	0.014 64	110	290	0.000 000 185
40	360	0.007 4	150	250	0.000 000 000 13
45	355	0.003 76	200	200	0.000 000 000 000 002 3
50	350	0.001 87	400	0	0.000 000 000 000 000 000

400 km above sea level at $r = 6,780$ km from the Earth center, a body, falling toward the Earth under gravity alone, has by (6) a velocity

$$v = \sqrt{2g_0 \frac{r_0^2}{r}} = \sqrt{2 \cdot 0.0098 \cdot \frac{6,380^2}{6,780}} = 10.9 \text{ km/sec.}$$

It is clear that such a velocity cannot be stopped within 400 km without damage to the vehicle and its passengers. The retardation distance can however be arbitrarily increased with a tangential approach.

A body, subject only to gravity of the Earth, moves at large distances radially or approximately parabolically, the Earth center being the focus of the motion, the velocity at distance r being

$$v = \sqrt{2g_0 \frac{r_0^2}{r}}$$

(using the notation of Figure 2), i.e., a tangential velocity, when passing the Earth, of

$$v_{max} = \sqrt{2g_0 r_0} = \sqrt{2 \cdot 0.0098 \cdot 6,380} = 11.2 \text{ km/sec,}$$

while, when passing higher up, at the limit of the atmosphere, the tangential velocity is

$$v = \sqrt{2 \cdot 0.0098 \cdot \frac{6,380^2}{6,780}} = 10.9 \text{ km/sec,}$$

thus the mean is

$$v' = 11.1 \text{ km/sec.}$$

To determine within which limits of the atmosphere a usable retardation may be obtained; Table IV shows the air resistance $w = \frac{\gamma v^2}{g}$, acting at right angles to the motion in kg/m^2 as a result of a velocity of 11.1 km/sec.

Table IV

$h - y$ km	y km	r km	$g = g_0 \dfrac{r_0^2}{r^2}$ m/sec²	$\gamma = \gamma_0\left(\dfrac{r}{h}\right)^{49}$ kg/m³	$w = \gamma \cdot \dfrac{v'^2}{g}$ kg/m²
400	0	6,780	8.69	0.000 000 000 000 000 000	0.000 000 000
200	200	6,580	9.21	0.000 000 000 000 000 002 3	0.000 000 03
150	250	6,530	9.36	0.000 000 000 000 13	0.001 7
110	290	6,490	9.48	0.000 000 000 185	2.4
105	295	6,485	9.50	0.000 000 000 423	5.5
100	300	6,480	9.51	0.000 000 000 98	12.7
95	305	6,475	9.53	0.000 000 002 2	28.5
90	310	6,470	9.54	0.000 000 004 9	63.4
85	315	6,465	9.56	0.000 000 010 6	137
80	320	6,460	9.57	0.000 000 023 0	297
75	325	6,455	9.59	0.000 000 049 7	640
70	330	6,450	9.60	0.000 000 102 5	1,320
65	335	6,445	9.62	0.000 000 217	2,780
60	340	6,440	9.63	0.000 000 448	5,720
55	345	6,435	9.65	0.000 000 915	11,800
50	350	6,430	9.66	0.001 870	23,900

The atmosphere above 100 km cannot be considered suitable for retardation for this velocity. On the other hand, one cannot expose the vehicle, which now has only its small final mass m -- contrary to the situation when penetrating the atmosphere from below, as investigated at the end of the last chapter -- and which is intentionally designed to utilize and not minimize friction to excessive resistances $w = \frac{\gamma v^2}{g}$, but rather with regard to maneuverability, things have to be so arranged, that as with an airplane, the velocity in the lower atmospheric strata, where g = 9.8 m/sec² and γ = 1.3 kg/m³, is 50 m/sec approximately, so that

$$w = \frac{\gamma v^2}{g} = \frac{1.3 \cdot 50^2}{9.8} \approx 330 \text{ kg/m}^2.$$

According to Table IV, a height between 75 and 100 km above the ground gives a good mean value.

The atmosphere should thus be entered with a tangential perigee of 75 km above sea level or

$$r_a = 6,380 + 75 = 6,455 \text{ km}$$

from the earth center.

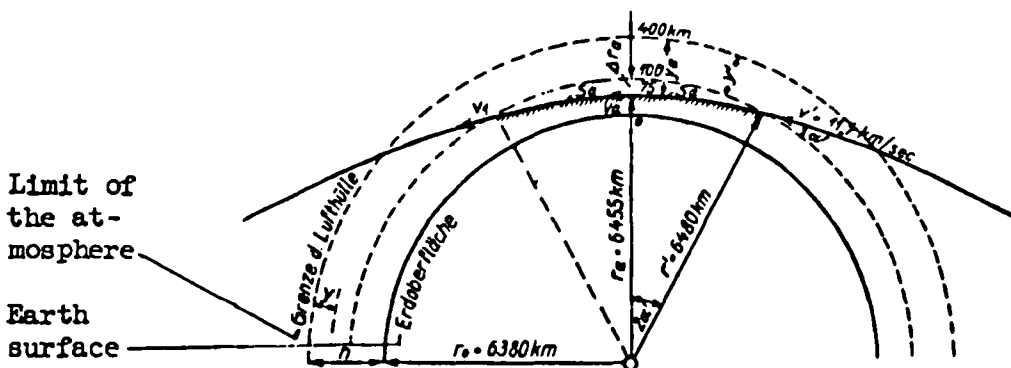The length of the retardation paths between 75 and 100 km, according to Figure 7:

Limit of the atmosphere

Earth surface

Figure 7

Generally for a parabola

$$\frac{r_a}{r^1} = \cos^2 \alpha',$$

i.e.,

$$\cos \alpha' = \sqrt{\frac{r_a}{r'}} = \sqrt{\frac{6,455}{6,480}} = 0.998975;$$

$$\alpha' = 3°34';$$

$$2\alpha' = 7°8';$$

and with sufficient accuracy

$$s_a = r' \sin 2\alpha' = 6,480 \cdot 0.12428 = 805 \text{ km};$$

i.e., between 75 and 100 km, the effective distance, during which retardation is experienced, is

$$2s_a = 1,610 \text{ km},$$

if in the first order we neglect changes in the trajectory due to retardation (these will be dealt with at the end of the section).

Within s_a the retardation β of the mass m_1, through the air resistance w, has the variable value

$$\beta = \frac{W}{m_1}$$

or (by (14) and (16a) with $g = \sim g_0$)

$$\frac{dv}{dt} = - \frac{\gamma_o^F \psi}{g_o m_1} \cdot v^2 \cdot \left(\frac{Y}{h}\right)^{49};$$

further

$$\frac{ds}{dt} = v$$

and approximately

$$\frac{ds}{dy} = \frac{s_a}{\Delta r_a} = \frac{s_a}{r' - r_a};$$

therefore

$$\frac{dv}{dy} = \frac{dv}{dt} \cdot \frac{dt}{ds} \cdot \frac{ds}{dy} = - \frac{\gamma_o^F \psi}{g_o m_1} \cdot \frac{s_a}{\Delta r_a} \cdot v\left(\frac{Y}{h}\right)^{49};$$

24

or

$$\frac{dv}{v} = -\frac{\gamma_o F\psi}{g_o m_1} \cdot \frac{s_a}{\Delta r_a} \cdot \left(\frac{y}{h}\right)^{49} \cdot dy;$$

$$\ln v = -\frac{\gamma_o F\psi}{50\, g_o m_1} \cdot \frac{s_a}{\Delta r_a} \cdot \frac{y^{50}}{h^{49}} + C;$$

when entering the retardation stretch, i.e., for $y = y'$ we have

$$\ln v' = -\frac{\gamma_o F\psi}{50\, g_o m_1} \cdot \frac{s_a}{\Delta r_a} \cdot \frac{y'^{50}}{h^{49}} + C;$$

in the center of the stretch, where $y = y_a$,

$$\ln v_a = -\frac{\gamma_o F\psi}{50\, g_o m_1} \cdot \frac{s_a}{\Delta r_a} \cdot \frac{y_a^{50}}{h^{49}} + C;$$

therefore during the passage of the first half:

$$\ln v' - \ln v_a = \ln \frac{v'}{v_a} = \frac{\gamma_o F\psi}{50\, g_o m_1} \cdot \frac{s_a}{\Delta r_a} \cdot h\, \left[\left(\frac{y_a}{h}\right)^{50} - \left(\frac{y'}{h}\right)^{50}\right] \quad (18)$$

If one uses the values:

$$\gamma_o = 1.3 \text{ kg/m}^3; \quad \Delta r_a = r' - r_a = 100 - 75 = 25 \text{ km};$$

$$s_a = 805 \text{ km}; \quad \frac{s_a}{\Delta r_a} = \frac{805}{25} = 32.2;$$

$$h = 400 \text{ km} = 400,000 \text{ m}; \quad y_a = 325 \text{ km}; \quad y' = 300 \text{ km};$$

and further as before uses $g_o m_1$ = weight of the vehicle G_1 on the earth = 2,000 kg and $F\psi = 6.1 \text{ m}^2$ corresponding say to a parachute of 2.8 m dia., moving at right angles to its area, so that the max. value of the retardation at 75 km above sea level is

$$\beta_{max} = \frac{w}{m_1} \cdot F\psi = \frac{640}{200} \cdot 6.1 = 19.5 \text{ m/sec}^2,$$

then one finds the velocity v_a at the perigee from

$$\ln \frac{v'}{v_a} = \frac{1.3 \cdot 6.1}{50 \cdot 2,000} \cdot 32.2 \cdot 400,000 \left[\left(\frac{325}{400}\right)^{50} - \left(\frac{300}{400}\right)^{50}\right] = 0.031,$$

or

$$\frac{v'}{v_a} = e^{0.031} = 1.032,$$

thus

$$v_a = \frac{v'}{1.032} \; .$$

Similarly for the second half s_a of the stretch we find the exit velocity

$$v_1 = \frac{v_a}{1.032} = \frac{v'}{1.032^2} = \frac{11.1}{1.032^2} = 10.4 \text{ km/sec.}$$

In consequence of the retardation, there occurs an alteration in the trajectory, in particular the parabola is replaced by an ellipse, so that the vehicle returns to the same retardation area after completing an orbit, but this time with an entrance velocity = $v_1 = 10.4$ km/sec. Since within the short breaking distance the elliptic orbit is only minutely different from the parabolic orbit, one may again use $2s_a = 2 \cdot 805 = 1,610$ km. After a second passage, the new velocity

$$v_2 = \frac{v_1}{1.032^2} = \frac{v'}{1.032^4} = \frac{11.1}{1.032^4} = 9.8 \text{ km/sec.}$$

This further lowering of velocity causes a smaller elliptic orbit, at the end of which a further breaking period follows with entrance velocity $v_2 = 9.8$ km/sec. If the same distance $2s_a = 1,610$ km is assumed (actually it increases a little every time, thus increasing the retardation), then

$$v_3 = \frac{11.1}{1.032^6} = 9.2 \text{ km/sec,}$$

and so on:

$$v_4 = \frac{11.1}{1.032^8} = 8.6 \text{ km/sec,}$$

$$v_5 = \frac{11.1}{1.032^{10}} = 8.1 \text{ km/sec,}$$

until finally, after a crossing of half the distance s_a, a velocity

$$v_a = \frac{v_5}{1.032} = \frac{11.1}{1.032^{11}} = 7.85 \text{ km/sec}$$

is reached. This however is the velocity

$$\sqrt{g_a r_a} = \sqrt{g_0 \frac{r_0^2}{r_a^2} \cdot r_a} = \sqrt{g_0 \frac{r_0^2}{r_a}} = \sqrt{0.0098 \cdot \frac{6,380^2}{6,455}} = 7.85 \text{ km/sec,}$$

corresponding to a circular orbit at the distance $r_a = 6,455$ km from the Earth center or 75 km above the Earth, providing friction is neglected. The vehicle would thus remain within the atmosphere, permitting a continuation of the landing in the form of a glide.

To calculate the time necessary for the passage through the various orbits during retardation, we first have to find their dimensions (see Figure 9).

A body at distance r from the Earth center E and of mass m experiences attraction

$$P = -\frac{\mu \cdot m}{r^2} .$$

At sea level, $r = r_0$, $P = mg_0$, i.e., the weight of the body

$$mg_0 = \frac{\mu \cdot m}{r_0^2} ,$$

so that

$$\mu = g_0 r_0^2 = 0.0098 \cdot 6,380^2 = 400,000 \text{ km}^3/\text{sec}^2.$$

If the body (Figure 8) has a velocity $v_a \perp r_a$ at distance r_a (perigee or apogee), then it describes an ellipse with semi-axis

$$a = \frac{\mu}{\dfrac{2\mu}{r_a} - v_a^2} \text{ and } b = \frac{v_a \cdot r_a}{\sqrt{\dfrac{2\mu}{r_a} - v_a^2}} .$$

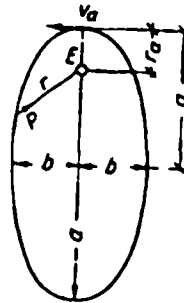(See derivation at the end of section III.)

Figure 8

If the corresponding exit velocities are assumed to occur with sufficient accuracy all at the same point r_a = 6,455 km, then approximately $\frac{2\mu}{r_a} = \frac{800,000}{6,455} = 124$:

for v_1 = 10.4 km/sec:

$$a_1 = \frac{400,000}{124 - 10.4^2} = 25,000 \text{ km},$$

$$b_1 = \frac{10.4 \cdot 6,455}{\sqrt{124 - 10.4^2}} = 16,800 \text{ km};$$

for v_2 = 9.8 km/sec:

$$a_2 = \frac{400,000}{124 - 9.8^2} = 14,300 \text{ km},$$

$$b_2 = \frac{9.8 \cdot 6,455}{\sqrt{124 - 9.8^2}} = 11,950 \text{ km};$$

for v_3 = 9.2 km/sec:

$$a_3 = \frac{400,000}{124 - 9.2^2} = 10,250 \text{ km},$$

$$b_3 = \frac{9.2 \cdot 6,455}{\sqrt{124 - 9.2^2}} = 9,500 \text{ km};$$

for v_4 = 8.6 km/sec:

$$a_4 = \frac{400,000}{124 - 8.6^2} = 8,000 \text{ km},$$

$$b_4 = \frac{8.6 \cdot 6,455}{\sqrt{124 - 8.6^2}} = 7,850 \text{ km};$$

for v_5 = 8.1 km/sec:

$$a_5 = \frac{400,000}{124 - 8.1^2} = 6,900 \text{ km},$$

$$b_5 = \frac{8.1 \cdot 6,455}{\sqrt{124 - 8.1^2}} = 6,860 \text{ km}.$$

The period of the orbits follows from the theorem (39), end of section III:

$$\frac{dF}{dt} = \text{constant} = \frac{v_a \cdot r_a}{2} \; ;$$

$$dF = \frac{v_a r_a}{2} \cdot dt;$$

$$F = \frac{v_a r_a}{2} \cdot t = ab\pi;$$

thus
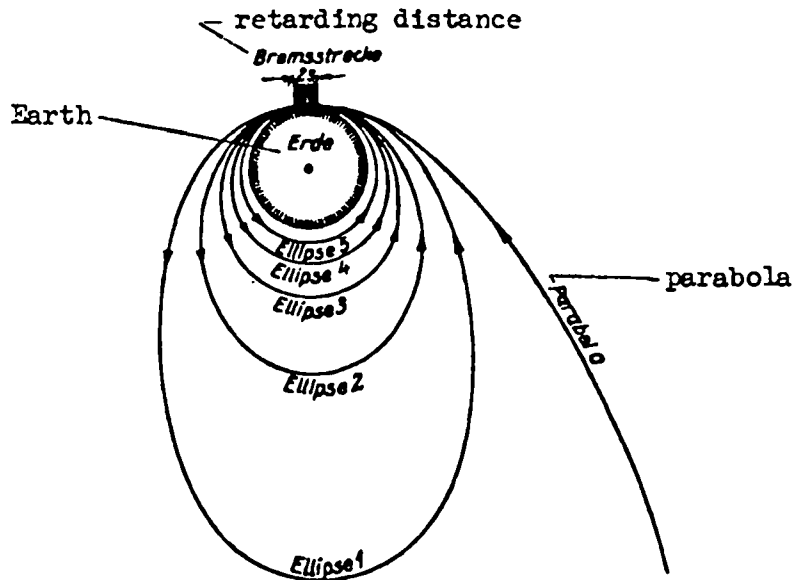
$$t = \frac{2ab\pi}{v_a \cdot r_a} \; . \tag{18a}$$

Figure 9

The time required for the 5 orbits is therefore:

$$t_1 = \frac{2 \cdot 25{,}000 \cdot 16{,}800 \cdot \pi}{10.4 \cdot 6{,}455} = 39{,}300 \text{ sec} = \sim 10.9 \text{ hours}$$

$$t_2 = \frac{2 \cdot 14{,}300 \cdot 11{,}950 \cdot \pi}{9.8 \cdot 6{,}455} = 16{,}900 \text{ sec} = \sim 4.7 \text{ hours}$$

$$t_3 = \frac{2 \cdot 10{,}250 \cdot 9{,}500 \cdot \pi}{9.2 \cdot 6{,}455} = 10{,}300 \text{ sec} = \sim 2.9 \text{ hours}$$

$$t_4 = \frac{2 \cdot 8{,}000 \cdot 7{,}850 \cdot \pi}{8.6 \cdot 6{,}455} = 7{,}100 \text{ sec} = \sim 2.0 \text{ hours}$$

$$t_5 = \frac{2 \cdot 6{,}900 \cdot 6{,}860 \cdot \pi}{8.1 \cdot 6{,}455} = 5{,}700 \text{ sec} = \sim 1.6 \text{ hours}$$

a total of $t_a = 79{,}300 \text{ sec} = \sim 22.1 \text{ hours}$

Now commences a glide, which may be thought of as follows: It begins at height $h - y_a = 75$ km, with a tangential velocity $v_a = 7.85$ km/sec, with the centrifugal force $z_a = \frac{v_a^2}{r_a}$, is exactly equal to g_a, since, according to page 25, $v_a^2 = g_a \cdot r_a$. Because of the resistance, a constant deceleration β acts and therefore v and with it the centrifugal force $z = \frac{v^2}{r}$, decrease steadily, while the gravitational force g remains about constant. In addition to the tangential retardation, there has thus to be provided a steadily increasing radial acceleration e to balance g versus a centrifugal force z:

$$e = g - z = g\left(1 - \frac{z}{g}\right),$$

or, since $z = \frac{v^2}{r}$ and, within the range of 0 and 75 km, with sufficient accuracy also $g = \frac{v_a^2}{r}$:

$$e = g\left(1 - \frac{v^2}{v_a^2}\right) \tag{19}$$

This radial acceleration e may be provided by a wing F_o, which is gradually brought to bear by moving its plane from horizontal toward an inclination as shown in Figure 10:
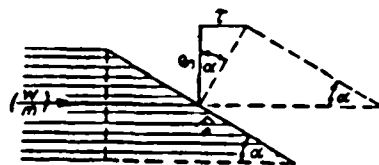

Figure 10

$$e = \frac{w}{m} \cdot \vec{r}_0 \cdot \sin^2 \alpha \cdot \cos \alpha; \qquad (20)$$

the simultaneous tangential component $\tau = e \cdot \tan \alpha$ may be still neglected with respect to the large deceleration β.

To maintain an equal maneuverability, the resistance w must not increase over its initial value, i.e., by (14) and (16a):

$$w = \frac{\gamma_0}{g_0} v^2 (\frac{y}{h})^{49} = \frac{\gamma_0}{g_0} v_a{}^2 (\frac{y_a}{h})^{49};$$

i.e., things have to be arranged so that always

$$\frac{v^2}{v_a{}^2} = \frac{(\frac{y_a}{h})^{49}}{(\frac{y}{h})^{49}} = (\frac{y_a}{y})^{49} \qquad (21)$$

applies. A definite height y may only be taken up, when the velocity v has been decreased correspondingly.

In Figure 11, the values of $\frac{v^2}{v_a{}^2}$ belonging to each height y, are shown together with the values of $1 - \frac{v^2}{v_a{}^2}$, which by (19) represent the required increase in e on a scale 1:g.
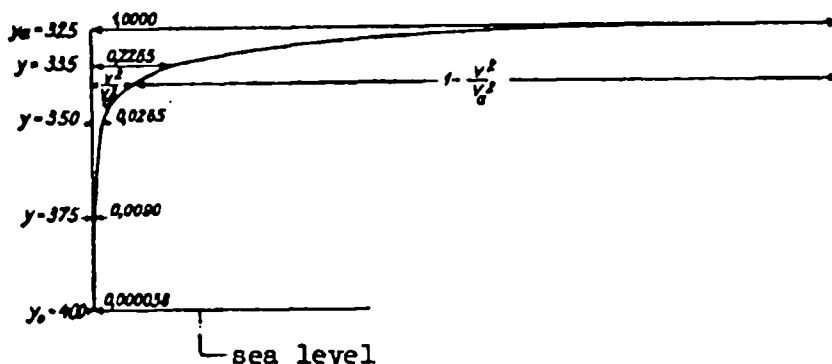

Figure 11.

Further the distance s, covered when a velocity v has been reached at constant deceleration $\beta = \beta_a$, is:

$$s = \frac{v_a^2 - v^2}{2\beta_a} = \frac{v_a^2}{2\beta_a}\left(1 - \frac{v^2}{v_a^2}\right) = \frac{v_a^2}{2\beta_a}\left[1 - \left(\frac{y_a}{y}\right)^{49}\right], \quad (22)$$

so that s also is given by $1 - \frac{v^2}{v_a^2}$ in Figure 11, on a scale $1:\frac{v_a^2}{2\beta_a}$.
From this it may be seen that, if β remained constant, the glide would turn from its initial favorable course into a crash. Thus may remain constant only to the point, where the trajectory begins to deviate more strongly from the horizontal.

Now, the inclination of the trajectory, according to (22), is

$$\frac{ds}{dy} = \frac{v_a^2}{2\beta_a} \cdot 49 \cdot \frac{y_a^{49}}{y^{50}} = \frac{v_a^2}{2\beta_a} \cdot \frac{49}{y_a}\left(\frac{y_a}{y}\right)^{50},$$

which gives

$$\left(\frac{y}{y_a}\right)^{50} = \frac{49}{y_a} \cdot \frac{v_a^2}{2\beta_a} \cdot \frac{dy}{ds}. \quad (23)$$

If the retardation corresponding to height $h - y_a = 75$ km ($y_a = 325$ km and $v_a = 7.85$ km/sec) resulting from $F = 6.1$ m^2 and given by

$$\beta_a = \frac{w}{m_1} \cdot F = \frac{\gamma_0}{g_0 m_1} \cdot v_a^2 \cdot \left(\frac{y_a}{h}\right)^{49} \cdot F = \frac{1.3}{2,000} \cdot 7,850^2 \cdot 6.1 \cdot$$

$$\left(\frac{325}{400}\right)^{49} = 9.3 \text{ m/sec}^2 = 0.0093 \text{ km/sec}^2$$

is maintained, then (23) gives a limiting inclination of $\frac{dy}{ds} = \frac{1}{10}$,
which occurs at the height

$$\left(\frac{y_b}{y_a}\right)^{50} = \frac{49}{325} \cdot \frac{7.85^2}{2 \cdot 0.0093} \cdot \frac{1}{10} = 50,$$

or

$$y_b = y_a \cdot 50^{\frac{1}{50}} = 325 \cdot 1.0814 = 352 \text{ km},$$

or at

$$h - y_b = 400 - 352 = 48 \text{ km}$$

above the ground, with a velocity v_b, according to (21):

$$\frac{v_b^2}{v_a^2} = (\frac{y_a}{y_b})49 = (\frac{y_a}{y_b})50 \cdot \frac{y_b}{y_a} = \frac{1.0814}{50} = 0.02163,$$

or

$$v_b = v_a \sqrt{0.02163} = 7.85 \cdot 0.147 = 1.15 \text{ km/sec}$$

and after a distance according to (22):

$$s_b = \frac{v_a^2}{2\beta_a}(1 - \frac{v_b^2}{v_a^2}) = \frac{7.85^2}{2 \cdot 0.0093}(1 - 0.02163) = 3,250 \text{ km}$$

and after a time

$$t_b = \frac{v_a - v_b}{\beta_a} = \frac{7,850 - 1,150}{9.3} = 720 \text{ sec.}$$

The radial deceleration is by (19)

$$\varrho_b = g(1 - \frac{v_b^2}{v_a^2}) = g(1 - 0.02163) = 0.97837 \cdot g,$$

or nearly equal to the total gravitational acceleration g and may be thought resulting from a wing F_0, obeying (20):

$$\varrho = \frac{w}{m_1} \cdot F_0 \cdot \sin^2 \alpha \cdot \cos \alpha = \sim g,$$

where, according to our assumptions, w still has the value

$$w = \frac{y_0}{g_0} \cdot v_a^2 (\frac{y_a}{h})49 = \frac{1.3}{9.8} \cdot 7,850^2 (\frac{325}{400})49 = \sim 310 \text{ kg/m}^2$$

so that

$$F_0 \cdot \sin^2 \alpha \cdot \cos \alpha = \frac{m_1 g}{w} = \sim \frac{2,000}{310} = 6.5 \text{ m}^2.$$

α should be chosen as small as possible, say max $\alpha = 20°$, to get a value $\tau = \varrho \cdot \tan \alpha$ not excessively large with respect to β_a, so that

$$\text{max } \tau = 0.364 \cdot 9.8 = 3.56 \text{ m/sec}^2$$

compared to

$$\beta_a = 9 \text{ m/sec}^2$$

and

$$F_o = \frac{6.5}{0.342^2 \cdot 0.940} = 59 \text{ m}^2 \ (\sim 5 \text{ m} \cdot 12 \text{ m}).$$

I.e., the angle α of the wing has to increase from $0°$ to $20°$ to the horizontal for a constant wing area $F_o = 59$ m² and constant breaking area $F = 6.1$ m², while the distance $s_b = 3,250$ km is covered and the height drops from $h - y = 75$ down to 48 km, in order to have the radial deceleration e increase from zero to g, while the velocity $v_a = 7,850$ km/sec decreases down to $v_b = 1,150$ km/sec as a result of the constant resistance $w = 310$ km/m² (see Figures 12A-B).

From the height $h - y_b = 48$ km, β must be decreased to avoid an excessive descent, say, by eliminating the parachute type breaking area F, leaving only the component obtained last $\tau = 3.56$ m/sec² $= 0.00356$ km/sec², produced by the wing for the further retardation. This value may also not be maintained to the end, since it would result in too steep a trajectory after a short time.
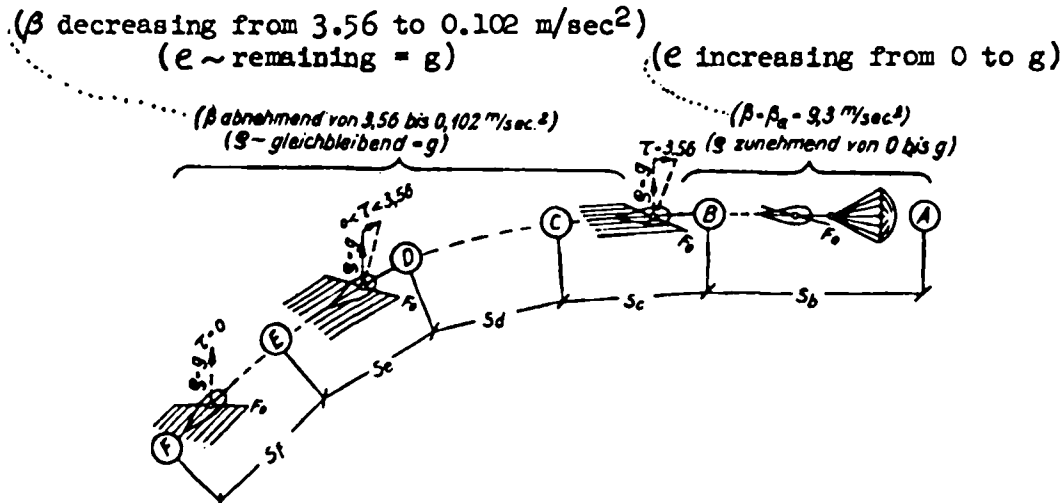
.(β decreasing from 3.56 to 0.102 m/sec²)
    ($e \sim$ remaining $= g$)   .(e increasing from 0 to g)

Figure 12.

If e remains constant (equal to g), then the retardation must be made to decrease, say, by changing the inclination of the wing F_o from B gradually to D and finally to the horizontal position at F (see Figure 12).

At all points of the trajectory:

$$- \beta ds = d\left(\frac{v^2}{2}\right)$$

or, since

$$v^2 = v_a{}^2 \cdot \left(\frac{y_a}{y}\right)^{49} :$$

$$- \beta ds = \frac{v_a{}^2}{2} \cdot d\left(\frac{y_a}{y}\right)^{49} = - \frac{v_a{}^2}{2} \cdot \frac{49}{y_a} \cdot \left(\frac{y_a}{y}\right)^{50} \cdot dy,$$

so that generally

$$\frac{ds}{dy} = \frac{v_a{}^2}{2\beta} \cdot \frac{49}{y_a} \cdot \left(\frac{y_a}{y}\right)^{50} \tag{24}$$

where β is constant.

If the flight is to terminate at $45°$, then for $y = y_0 = 400$ km, we must have:

$$\frac{dy}{ds} = \frac{1}{\sqrt{2}} \text{ (see Figure 13),}$$
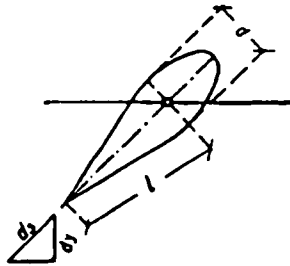
Figure 13.

the final value of β therefore is:

$$\beta_{min} = \frac{v_a{}^2}{2} \cdot \frac{49}{y_a} \cdot \left(\frac{y_a}{y_0}\right)^{50} \cdot \frac{dy}{ds} = \frac{7.85^2}{2} \cdot \frac{49}{325} \cdot \left(\frac{325}{400}\right)^{50} \cdot \frac{1}{\sqrt{2}} =$$

$$0.000102 \text{ km/sec}^2 = 0.102 \text{ m/sec}^2.$$

At the end of the trajectory (Figure 12F), the tangential component τ of the wing is zero and the retardation β_{min} is now solely caused by the atmospheric resistance against the nose cone, the form of which with reference to Figure 13 is thus

$$\beta_{min} = \frac{w}{m_1} \cdot \frac{d^2 \pi}{4} \cdot (\frac{d}{21})^2 \; ;$$

i.e.,

$$1 = \frac{d^2}{4} \cdot \sqrt{\frac{w \cdot \pi}{m_1 \cdot \beta_{min}}} \; ,$$

or putting in the values

$w = 310$ km/m^2 (according to our still valid assumption);

$$m_1 = \frac{2,000 \text{ kg}}{9.8 \text{ m/sec}^2} = \sim 200 \; \frac{\text{kg} \cdot \text{sec}^2}{m} \; ;$$

$d = 1.5$ (practical minimum dimensions of the vehicle):

$$1 = \frac{1.5^2}{4} \sqrt{\frac{310 \cdot \pi}{200 \cdot 0.102}} = 3.88 \text{ m.}$$

The final speed is given by

$$\frac{v^2}{v_a^2} = (\frac{325}{400})^{49} \; :$$

$$v = v_a \cdot (\frac{325}{400})^{\frac{49}{2}} = 7,850 \cdot 0.062 = 48.5 \text{ m/sec,}$$

so that actually the resistance is

$$w = \frac{\gamma_o}{g_o} v^2 = \frac{1.3}{9.8} \cdot 48.5^2 = 310 \text{ kg/m}^2$$

permitting a simple landing.

For simplicity, a stepwise reduction in deceleration replaces the gradual decrease from $\beta = 3.56$ to $\beta = 0.102$ m/sec^2. We assume this to take place in 4 steps: B-C, C-D, D-E and E-F (see Figure 12), with β_c, β_d, β_e and β_f being 3.5, 1.0, 0.2 and 0.102 m/sec^2 respectively. These are to lead to trajectory inclinations $\frac{dy}{ds} = \frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{1}{\sqrt{2}}$, there follows from these assumptions at the terminations of the respective sections:

Section B-C:

by equation (24):

$$\frac{ds}{dy} = \frac{v_a^2}{2\beta_c} \cdot \frac{49}{y_a} \cdot (\frac{y_a}{y_c})50$$

or

$$(\frac{y_c}{y_a})50 = \frac{v_a^2}{2\beta_c} \cdot \frac{49}{y_a} \cdot \frac{dy}{ds} = \frac{7.85^2}{2 \cdot 0.0035} \cdot \frac{49}{325} \cdot \frac{1}{6} = 222;$$

therefore

$$y_c = y_a \cdot 222^{\frac{1}{50}} = 325 \cdot 1.114 = 362 \text{ km}; \quad h - y_c = 38 \text{ km};$$

further by (21):

$$\frac{v_c^2}{v_a^2} = (\frac{y_a}{y_c})49 = \frac{1.114}{222} = 0.00502;$$

$$v_c = v_a \sqrt{0.00502} = 7.85 \cdot 0.0706 = 0.555 \text{ km/sec};$$

and by (22):

$$s_c = \frac{v_b^2 - v_c^2}{2\beta_c} = \frac{1.15^2 - 0.555^2}{2 \cdot 0.0035} = 146 \text{ km};$$

and

$$t_c = \frac{v_b - v_c}{\beta_c} = \frac{1,150 - 555}{3.5} = 170 \text{ sec};$$

Section C-D:

$$(\frac{y_d}{y_a})50 = \frac{v_a^2}{2\beta_d} \cdot \frac{49}{y_a} \cdot \frac{dy}{ds} = \frac{7.85^2}{2 \cdot 0.001} \cdot \frac{49}{325} \cdot \frac{1}{3} = 1,550;$$

$$y_d = y_a \cdot 1,550^{\frac{1}{50}} = 325 \cdot 1,158 = 377 \text{ km}; \quad h - y_d = 23 \text{ km};$$

$$\frac{v_d^2}{v_a^2} = (\frac{y_a}{y_d})49 = \frac{1.158}{1,550} = 0.00075;$$

$$v_d = 7.85 \sqrt{0.00075} = 0.215 \text{ km/sec}$$

$$s_d = \frac{v_c^2 - v_d^2}{2\beta_d} = \frac{0.555^2 - 0.215^2}{2 \cdot 0.001} = 131 \text{ km};$$

$$t_d = \frac{v_c - v_d}{\beta_d} = \frac{555 - 215}{1} = 340 \text{ sec};$$

Section D-E:

$$(\frac{y_e}{y_a})50 = \frac{v_a^2}{2\beta_e} \cdot \frac{49}{y_a} \cdot \frac{dy}{ds} = \frac{7.85^2}{2 \cdot 0.0002} \cdot \frac{49}{325} \cdot \frac{1}{2} = 11,600;$$

$$y_e = y_a \cdot 11,600^{\frac{1}{50}} = 325 \cdot 1.206 = 392 \text{ km}; \quad h - y_e = 8 \text{ km};$$

$$\frac{v_e^2}{v_a^2} = (\frac{y_a}{y_e})49 = \frac{1.206}{11,600} = 0.000104;$$

$$v_e = 7.85 \sqrt{0.000104} = 0.080 \text{ km/sec};$$

$$s_e = \frac{v_d^2 - v_e^2}{2\beta_e} = \frac{0.215^2 - 0.080^2}{2 \cdot 0.0002} = 99 \text{ km};$$

$$t_e = \frac{v_d - v_e}{\beta_e} = \frac{215 - 80}{0.2} = 675 \text{ sec};$$

Section E-F:

$$y = 400 \text{ km}; \quad h - y = 0; \quad v_f = \sim 49 \text{ m/sec};$$

$$s_f = \frac{v_e^2 - v_f^2}{2\beta_f} = \frac{0.080^2 - 0.049^2}{2 \cdot 0.0001} = 20 \text{ km};$$

$$t_f = \frac{v_e - v_f}{\beta_f} = \frac{80 - 49}{0.1} = 310 \text{ sec}.$$

The total gliding distance is therefore

$$s_{b-f} = 3,250 + 146 + 131 + 99 + 20 = 3,646 \text{ km}$$

and lasts

$$t_{b-f} = 720 + 170 + 340 + 675 + 310 = 2,215 \text{ sec} = \sim 37 \text{ min}.$$

The entire landing period from the first entrance into the atmosphere to the final touch-down is

$$79,300 + 2,200 = 81,500 \text{ sec} = \sim 22.6 \text{ hours}.$$

In the treatment of the breaking ellipses it was assumed that to first order there is a sudden change of the orbit from one ellipse (or parabola) to the next. Actually this change is gradual because of the gradual resistive action and the change occurs in a transition spiral. The transition forces the vehicle into lower and therefore denser layers, which in turn cause a greater friction than was assumed. In consequence, the true exit ellipse has an inclined as well as shortened axis. To get an idea of the possible error, we shall in the following determine the transition spiral between the entrance parabola and the first ellipse by fitting elliptic sections together.

To this end one may divide the angle $4\alpha' = 14°16'$, which is the extent of the parabola within the effective layers into six sections, each of $\Delta\varphi = 2°22\ 2/3'$, each of which covers a distance of $\Delta s = \dfrac{1,610}{6} = \sim 270$ km on the assumed transition spiral. As required, further angles may be added to those on the left of Figure 7. At the termination of each section the retarding action of the following section Δs is assumed to occur in a sudden decrease of velocity $\Delta v = \dfrac{\beta \cdot \Delta s}{v}$, where v is the velocity for the preceding section and β may be found from table IV through the relation $\beta = \dfrac{w}{m_1} \cdot F \cdot (\dfrac{v}{v'})^2$. Values, not directly given, have been linearily extrapolated for better comparison. For the beginning of each section the values r_1, v_1, α_1 are given by Δv and an investigation of the preceding section and from these there follows by means of the equations

$$a = \frac{\mu}{\dfrac{2\mu}{r_1} - v_1^2} \ ; \ b^2 = \frac{v_1^2 r_1^2 \cos^2\alpha_1}{\dfrac{2\mu}{r_1} - v_1^2} \ ; \ \mu = g_0 r_0^2 ;$$

(c.f. (45) and (46) in connection with the area theorem) and

$$\cos\varphi_1 = \frac{\dfrac{b^2}{r_1} - a}{\sqrt{a^2 - b^2}}$$

(c.f. equation of an ellipse)

the angle φ_1 between the initial direction and the corresponding major axis of the elliptic section considered; further, since $\Delta\varphi = 2°22\ 2/3'$ is known, the angle $\varphi_2 = \varphi_1 + \Delta\varphi$ between the final direction and the major axis a and finally the values

$$r_2 = \frac{b^2}{a + \sqrt{a^2 - b^2} \cdot \cos \varphi_2}$$

(c.f. equation of an ellipse);

$$v_2 = \sqrt{\frac{2\mu}{r_2} - \left(\frac{2\mu}{r_1} - v_1^2\right)}$$

(see equation (4))

and

$$\cos \alpha_2 = \cos \alpha_1 \cdot \frac{r_1 v_1}{r_2 v_2}$$

(see area theorem, (39)),

belonging to the end points of an elliptic sector. This is repeated until again a distance $r > 6,480$ km is reached, being the entrance ray of the exit ellipse.

The trajectory elements, calculated in this manner, are compiled in the table on page 40.

Sector	0	I	II	III	IV	V	VI	VII
r_1 (km)	6,480	6,480	6,466	6,457	6,454	6,456	6,462	6,472
v_1 (km/sec)	11.10	11.09	11.00	10.66	10.20	9.80	9.60	9.57
α_1	3°34'	3°34'	2°22⅔'	1°17'	>0°0', <0°16'	>0°55', <0°59'	1°45'	2°30'
$a = \dfrac{2\mu}{r_1} - v_1^2$	(Parabola)	←	148,030	38,987	20,080	14,347	12,641	12,486
$b^2 = \dfrac{(v_1 r_1 \cos \alpha_1)^2}{\frac{2}{r_1} - v_1^2}$	(Parabola)	←	1,869.1 × 10⁶	461.56 × 10⁶	217.525 × 10⁶	143.570 × 10⁶	121.500 × 10⁶	119.500 × 10⁶
$e = \sqrt{a^2 - b^2}$		-------	141,580	32,534	13,627	7,894	6,188	6033.3
φ_1 from $\cos \varphi_1 = \dfrac{\frac{b^2}{r_1} - a}{e}$		7°8'	5°4'	2°50'	~0°	2°30'	5°15'	7°41'
$\Delta\varphi$		2°22⅔'	2°22⅔'	2°22⅔'	2°22⅔'	2°22⅔'	2°22⅔'	2°22⅔'
$\varphi_2 = \varphi_1 \pm \Delta\varphi$		4°45⅓'	2°41⅓'	0°27⅓'	2°22⅔'	4°52⅔'	7°37⅔'	10°2⅔'
$r_2 = \dfrac{b^2}{a + e \cos \varphi_2}$	6,480	6,466	6,457	6,454	6,456	6,462	6,472	6,485
$v_2 = \sqrt{\dfrac{2}{r_2} - \left(\dfrac{2}{r_1} - v_1^2\right)}$	11.10	11.10	11.01	10.663	10.198	9.794	9.5905	-------
a_2 from $\cos a_2 = \cos a_1 \times \dfrac{r_1 v_1}{r_2 v_2}$	3°34'	2°22⅔'	1°17'	>0°0', <0°16'	>0°55', <0°59'	1°45'	2°30'	-------
$\beta = \dfrac{v}{n_1} \times F \times \left(\dfrac{v_2}{v}\right)^2$	0.00038	0.00412	0.014	0.018	0.016	0.0067	0.00064	-------
Δs	270	270	270	270	270	270	270	-------
$\Delta v = \dfrac{\beta \, \Delta s}{v_2}$	0.01	0.10	0.35	0.46	0.40	0.19	0.018	-------
$v_2 = \Delta v$	~11.09	~11.00	~10.66	~10.20	~9.80	~9.60	~9.57	-------

For comparison, the values corresponding to a sudden transition from the parabola to the first ellipse are shown below in contrast to those due to gradual transition:

Limiting Point		O-I	I-II	II-III	III-IV
Parabola and first ellipse	r	6,480	6,466	6,458	6,455
	v	11.10	11.11	11.12	10.40
	$\propto$	$3°34'$	$2°22\ 2/3'$	$1°11\ 1/3'$	$0°0'$
Spiral	r	6,480	6,466	6,457	6,454
	v	11.10	11.00	10.66	10.20
	$\propto$	$3°34'$	$2°22\ 2/3'$	$1°17'$	$>0°0'$ $<0°16'$

Limiting Point		IV-V	V-VI	VI-VII	VII-VIII
Parabola and first ellipse	r	6,457.5	6,464.5	6,476.3	--
	v	10.40	10.39	10.38	--
	$\propto$	$1°0'$	$2°1'$	$3°2'$	--
Spiral	r	6,456	6,462	6,472	6,485
	v	9.80	9.60	9.57	--
	$\propto$	$>0°55'$ $<0°59'$	$1°45'$	$2°30'$	--

The resulting ellipse accordingly has an a = 12,486 km in place of 25,000 km and b = $\sqrt{119,500,000}$ = 10,931 km in place of 16,800 km and is thus much smaller than the first ellipse calculated previously; the major axes being displaced by $7°41'$ - $7°8'$ = 33' from that of the former. Perigee would be at

$$r_a = \frac{b^2}{a + e} = \frac{119,500,000}{12,486 + 6,033} = 6,452.7 \text{ km}$$

in place of 6,455 km.

This makes it probable that actually a passage through two ellipses, instead of the earlier estimate of five, will be

sufficient to attain the correct orbitting velocity, particularly, if the area F is slightly increased.

Finally, it is to be investigated, whether one could not force a circular orbit at the first entrance into the atmosphere, without employing ellipses. This requires altitude control. Since this however is required anyway for the subsequent glide, this is no handicap.

The first, unfavorable approximations give a vertex of r_a = 6,455 km for the parabola with an already diminished velocity of $v_a = \dfrac{11.1}{1.032}$ = 10.75 km/sec. If the vehicle is to be forced into a circular orbit with this velocity and at this distance, a centripetal acceleration

$$z_a = \frac{v_a{}^2}{r_a} = \frac{10,750^2}{6,455,000} = 17.9 \ \text{m/sec}^2$$

is required there in place of the gravitational acceleration of

$$g_a = 9.8 \cdot \left(\frac{6,380}{6,455}\right)^2 = 9.6 \ \text{m/sec}^2.$$

The corresponding radial added acceleration

$$e = z_a - g_a = 8.3 \ \text{m/sec}^2$$

may be produced through the effect of the air resistance on the wing F_o, which, in accordance with Figure 13a, has to be placed at an angle α with respect to horizontal, so that by (20):

$$e = \frac{w}{m} \cdot F_o \cdot \sin^2 a \cdot \cos \alpha.$$
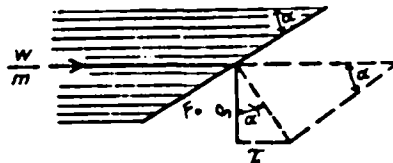
Figure 13a.

With decreasing trajectory velocity v, the required radial acceleration e slowly decreases, which may be achieved by a corresponding decrease in the angle α.

For $v_a = 10.75$ km/sec and $r_a = 6,455$ km we have, maintaining F_o at the same area as previously, namely 59 m^2 and a vehicle mass

$$m = \sim \frac{2,000 \text{ kg}}{10 \text{ m/sec}^2} = 200 \frac{\text{kg} \cdot \text{sec}^2}{\text{m}} :$$

$$w = 640 \cdot (\frac{10.75}{11.10})^2 = 600 \text{ kg/m}^2$$

and

$$\frac{w}{m} \cdot F_o = \frac{600 \text{ kg/m}^2}{200 \frac{\text{kg} \cdot \text{sec}^2}{\text{m}}} \cdot 59 \text{ m}^2 = 177 \text{ m/sec}^2;$$

therefore for circular orbit we must have

$$\sin^2 \alpha \cdot \cos \alpha = \frac{e}{\frac{w}{m} \cdot F_o} = \frac{8.3}{177} = 0.047;$$

$$\alpha = \sim 12 \ 2/3°.$$

The angle α is gradually to be decreased to zero°, when the free orbitting velocity 7.85 km/sec is reached.

The retardation at 75 km height, where $v_{max} = 11.1$ km/sec and for the parachute area $F = 6.1$ m^2 was previously given as $\beta_{max} = 0.0193$ km/sec^2. During the forced circular orbit at 75 km height, the retardation for an instantaneous velocity v:

$$\beta = \frac{dv}{dt} = -v^2 \cdot \frac{\beta_{max}}{v^2_{max}} = -v^2 \cdot k, \quad (\text{wo } k = \frac{0.0193}{11.1^2});$$

is therefore, further

$$\frac{ds}{dt} = v;$$

therefore

$$\frac{dv}{ds} = -vk;$$

$$kds = -\frac{dv}{v};$$

$$-ks = \ln v + C$$

44

at the parabola vertex for s = 0:

$$0 = \ln v_a + C; \quad C = - \ln v_a;$$

therefore

$$- ks = \ln v - \ln v_a = \ln \frac{v}{v_a} ;$$

or

$$s = \frac{1}{k} \cdot \ln \frac{v}{v_a} .$$

Accordingly, we have, at the end of the forced orbit, i.e., at v = 7.85 km/sec, which corresponds to the free circular orbit, covered a total distance

$$\text{max } s = \frac{11.1^2}{0.0193} \cdot \ln \frac{1,075}{785} = 6,400 \cdot (6.98008 - 6.66568) = 2,000 \text{ km.}$$

The time required to cover this distance follows from

$$\frac{dv}{dt} = - v^2 \cdot k;$$

$$kdt = - \frac{dv}{v^2} ;$$

$$kt = + \frac{1}{v} + C;$$

for t = 0, i.e., at the vertex of the parabola:

$$0 = \frac{1}{v_a} + C; \quad C = - \frac{1}{v_a} ;$$

therefore

$$kt = \frac{1}{v} - \frac{1}{v_a} ;$$

and

$$t = \frac{1}{k} \cdot (\frac{1}{v} - \frac{1}{v_a}) = \frac{1}{\beta_{max}} \cdot (\frac{v_{max}^2}{v} - \frac{v_{max}^2}{v_a});$$

$$t = \frac{1}{0.0193} \left(\frac{11 \cdot 10^2}{7.85} - \frac{11 \cdot 10^2}{10.75} \right) = \frac{15.7 - 11.5}{0.0193} = 218 \text{ sec} = 3.63 \text{ min.}$$

The entire landing time, including the subsequent glide, is only

$$218 + 2,200 = \sim 2,400 \text{ sec} = 40 \text{ min.}$$

A landing without breaking ellipses is therefore very well possible. However, the forced orbit, during which the passengers, because of centrifugal force, are pressed against the upper wall, represents an inverse flight, during which safety of maneuverability is perhaps impaired. The pilot however will have to see to it that he does not get into too low a strata, since this, according to Figure 11, could lead to a crash. If he, however, remains too high, then he will in the worst case bring his vehicle out of the atmosphere temporarily and enter a smaller or larger elliptical orbit, after which he can again, duly relaxed, attempt a landing.

In apparent contradiction to the proposed landing method, there stands the fact that meteorites burn up when entering the earth atmosphere, indicating strong frictional heating. Here it may be replied that these meteorites have a much higher entrance velocity than our vehicle. We have after all specified that it is subject only to the attraction of the earth indirectly, therefore that it is following the movement of the earth around the sun, which amounts to 30 km/sec, representing the unavoidable attraction of the sun, while meteorites, because of the gravitational field of the sun, generally reach the Earth's orbit at 42 km/sec. To this must be added the velocity of the Earth, 30 km/sec, when the velocity vectors are opposed, so that in this unfavorable case a velocity of 42 + 30 = 72 km/sec results, compared to the 11.1 km/sec of our vehicle. Since the resistive forces are proportional to the square of the velocity, we have in this unfavorable case a resistance $\left(\frac{72}{11} \right)^2 = 43$ times as large as that against the vehicle. Of course, it must not be overlooked that during the retardation of $v' = 11,100$ m/sec to $v = 0$ an energy $\frac{mv'^2}{2}$ becomes available. This for our assumed mass of approx.

$$m = \frac{2,000 \text{ kg}}{10 \text{ m/sec}^2} = 200 \frac{\text{kg} \cdot \text{sec}^2}{\text{m}}$$

gives

$$\frac{mv'^2}{2} = \frac{200}{2} \cdot 11,100^2 = 12,300,000,000 \text{ mkg.}$$

This energy must either be transformed into turbulence or heat or both. Up to now, we indirectly assumed transformation into turbulence. The other extreme -- total transformation into heat -- using a mechanical equivalent $\frac{1}{427}$, would lead to a value for the heat

$$Q = \frac{12,300,000,000}{427} = 28,800,000 \text{ heat units.}$$

During the retardation, so far assumed as high as possible, the parachute would become hot and burn up. It would therefore be necessary to bring along for the several passes through the retarding range and for the glide up to point B in Figure 12 a whole series of suitable parachutes to be used one after the other. (Since at point B the vehicle is already down to 1,150 m/sec, a further heating need not be feared.)

If however any kind of combustion is to be avoided, retardation would have to be sufficiently reduced, so that heated surfaces would have sufficient time to radiate or conduct the generated heat to the surroundings.

Generally, the energy becoming available between a given velocity v' to the instantaneous velocity v is

$$E = \frac{mv'^2}{2} - \frac{mv^2}{2} ;$$

its increase/sec thus:

$$\frac{dE}{dt} = mv \cdot \frac{dv}{dt} ;$$

the corresponding heat/sec therefore

$$\frac{dQ}{dt} = \frac{mv}{427} \cdot \frac{dv}{dt} ;$$

or, if the permissible heat intake/sec $\frac{dQ}{dt}$ is known, then the retardation for an instantaneous velocity v can at most be

$$\frac{dv}{dt} = \frac{dQ}{dt} \cdot \frac{427}{mv} .$$

The heat intake/sec is equal to the rate of transfer through radiation and conduction and may, if necessary, by means of cooling rims on the surface of the vehicle, be assumed to be 500 heat units/sec, so that, using again $m = 200 \frac{kg \cdot sec^2}{m}$:

$$\frac{dv}{dt} = \frac{500 \cdot 427}{200 \cdot v} = \sim \frac{1,000}{v} ; \quad (v \text{ in m/sec})$$

f.i., the maximum retardation can be

for $v = 10,000$ m/sec: $\frac{dv}{dt} = \frac{1,000}{10,000} = 0.1$ m/sec^2,

" $v = 5,000$ " $\frac{dv}{dt} = \frac{1,000}{5,000} = 0.2$ " ,

" $v = 1,000$ " $\frac{dv}{dt} = \frac{1,000}{1,000} = 1.0$ " ,

" $v = 100$ " $\frac{dv}{dt} = \frac{1,000}{100} = 10.0$ " .

To produce such a small deceleration a parachute is hardly necessary, the air resistance against the body and wings alone should suffice.

The entire landing distance s is now composed as follows:

$$\left.\begin{array}{l} \frac{dv}{dt} = \frac{1,000}{v} \\[2mm] \frac{ds}{dt} = v \end{array}\right\} \text{ therefore } \frac{dv}{ds} = \frac{1,000}{v^2}$$

$$ds = \frac{v^2 dv}{1,000} ;$$

$$s = \frac{1}{1,000} \cdot \int_0^{11,100} v^2 dv = \frac{11,100^3}{3 \cdot 1,000} = 410,700,000 \text{ m} = 410,700 \text{ km} =$$

approximately 10 Earth circumferences!

Of these the travel between $v = 11,100$ and $7,850$ m/sec (forced circular orbit) takes up:

$$\frac{11,100^3 - 7,850^3}{3 \cdot 1,000} = 249,450,000 \text{ m} = \text{approx. 6 Earth circumferences;}$$

between v = 7,850 and 4,000 m/sec:

$$\frac{7,850^3 - 4,000^3}{3 \cdot 1,000} = 139,920,000 \text{ m} = \text{approx. 3.5 Earth circumferences;}$$

between v = 4,000 and 0 m/sec:

$$\frac{4,000^3}{3 \cdot 1,000} = 21,330,000 \text{ m} = \text{approx. 0.5 Earth circumferences.}$$

All this under the incorrect assumption, that the entire breaking energy is transformed into heat.

The reality lies between the two extremes considered. In any case, the following has to be taken into consideration during landing:

1. Retardation must not be too strong, i.e., the parachute must not be chosen too large.

2. The parachute must have a form suitable for the formation of turbulence (the requirements 1 and 2 are best met by following Valier's suggestion of replacing the parachute by a number of concentrically arranged cones, whose vertices are directed forward).

3. Because of the possibility of combustion, a number of spare parachutes (or cones) has to be taken along.

4. The vehicle must not only be provided with wings, but also with cooling fins, made of metal.

Furthermore, these conditions being subject to such unusually high velocities and such unusually low atmospheric densities, require further experimental clarification.

III

FREE-SPACE TRAVEL

In the sections so far we have considered departure from the Earth up to the velocity, where no return to Earth occurs, and the return to Earth from the moment of entering the atmosphere, separately. The question now arises whether, after the Earth has been left behind, the vehicle can be directed in such a manner, that a return in the desired manner, i.e., tangentially, is possible.

After acceleration ceases, the vehicle moves in a radial direction away from Earth, providing we ignore for the sake of simplicity the lateral initial velocity due to the Earth rotation (approx. 463 m/sec at the equator). It rises or falls with a steady decreasing velocity into space and beyond doubt, the passengers will with the sudden cessation of gravity first of all sense in all probability the fear of a steady fall, which after some experience will go over into the more pleasant feeling of floating. Whether the velocity zero is finally reached at infinity depends on the highest velocity v_1, reached at the distance r_1, when acceleration ceases, which after all has not been exactly determined because of the uncertain air resistance. In any case, let the velocity, to be determined at an arbitrary distance r_2 from the center of the earth by means of distance measurements taken in definite time intervals, be v_2'.

Generally, the retardation due to gravity at the distance r is

$$\frac{dv}{dt} = - g_o \frac{r_o^2}{r^2}$$

and the velocity

$$\frac{dr}{dt} = v;$$

therefore

$$\frac{dv}{dr} = - \frac{g_o r_o^2}{r^2 v},$$

or

$$v dv = - g_o r_o^2 \frac{dr}{r^2},$$

from which

$$\frac{v^2}{2} = + \frac{g_o r_o^2}{r} + C;$$

thus at the distance r_2:

$$\frac{v_2'^2}{2} = \frac{g_o r_o^2}{r_2} + C;$$

and therefore

$$\frac{v_2'^2 - v^2}{2} = \frac{g_o r_o^2}{r_2} - \frac{g_o r_o^2}{r} \; ; \tag{25}$$

from this the distance r_3', at which velocity becomes zero, follows

$$\frac{v_2'^2}{2} = \frac{g_o r_o^2}{r_2} - \frac{g_o r_o^2}{r_3'} = g_o r_o^2 \left(\frac{1}{r_2} - \frac{1}{r_3'}\right); \tag{25a}$$

$$r_3' = \frac{2g_o r_o^2}{\dfrac{2g_o r_o^2}{r_2} - v_2'^2} \; . \tag{26}$$

If the height is to be not r_3' but r_3 then at the point r_2 the velocity would have to be in accordance with (25a):

$$v_2 = \sqrt{2g_o r_o^2 \left(\frac{1}{r_2} - \frac{1}{r_3}\right)} = \sqrt{2g_o r_o^2 \cdot \frac{r_3 - r_2}{r_2 r_3}} \tag{27}$$

in place of v_2'.

The velocity v_2' must therefore be corrected by $\Delta v_2 = v_2 - v_2'$. This can be done by firing a directional shot of mass Δm with a velocity c from the present vehicle mass m, so that according to (1):

$$\frac{\Delta m}{m} = \frac{\Delta v_2}{c}$$

with a $\pm$ sign, according to whether Δv is positive or negative.

If f.i. at the distance $r_2 = 40,000$ km the measured velocity is $v_2' = 4.46$ km/sec (which would give $r_3' = \infty$), and if r_3 is to be only 800,000 km (say about twice the distance to the moon), then according to (27) with

$$2g_o r_o^2 = 2 \cdot 0.0098 \cdot 6,380^2 = 800,000 \text{ km}^3/\text{sec}^2$$

we must have:

$$v_2 = \sqrt{2g_o r_o^2 \cdot \frac{r_3 - r_2}{r_2 r_3}} = \sqrt{800,000 \cdot \frac{800,000 - 40,000}{40,000 \cdot 800,000}} = 4.35 \text{ km/sec,}$$

therefore

$$\Delta v_2 = v_2 - v_2' = 4.35 - 4.46 = -0.11 \text{ km/sec,}$$

and for a velocity of the projectile of c = 1.0 km/sec:

$$\frac{\Delta m}{m} = \frac{0.11}{1.0} = 0.11;$$

i.e., a shell of approx. 1/9 of the present vehicle mass m would have to be fired in the direction of travel with a velocity of 1,000 m/sec. This shot is the more effective, the sooner it is fired.

After reaching the desired height r_3, the vehicle, if left to itself, would return to earth radially. In order to achieve the tangential entrance into the atmosphere, required in accordance with section II, it must receive at the point r_3, when the rate of velocity is zero, a tangential velocity v_3 (see Figure 14). The return trajectory will in that case however not be a parabola as provided in section II, but a very elongated ellipse of major axis
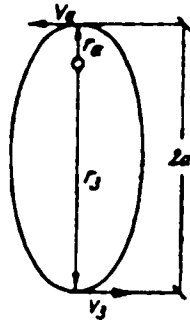
Figure 14.

$$a = \frac{r_3 + r_a}{2} ;$$

on the other hand however, by the law of gravity (see (45) at the end of this section):

$$a = \frac{g_0 r_0^2}{\frac{2 g_0 r_0^2}{r_3} - v_3^2} ;$$

therefore

$$\frac{g_0 r_0^2}{\frac{2 g_0 r_0^2}{r_3} - v_3^2} = \frac{r_3 + r_a}{2} ;$$

from which

$$v_3{}^2 = \frac{2g_0r_0{}^2}{r_3} - \frac{2g_0r_0{}^2}{r_3 + r_a} = 2g_0r_0{}^2 \cdot \frac{r_a}{r_3(r_3 + r_a)} \; ;$$

or

$$v_3 = \sqrt{2g_0r_0{}^2 \, \frac{r_a}{r_3(r_3 + r_a)}} \; ; \qquad\qquad (28)$$

and similarly

$$v_a{}^2 = 2g_0r_0{}^2 \cdot \frac{r_3}{r_a(r_3 + r_a)} = v_3{}^2 \cdot \frac{r_3{}^2}{r_a{}^2} \; ,$$

or

$$v_a = v_3 \cdot \frac{r_3}{r_a} \; .$$

F.i. for r_3 = 800,000 km; r_a = 6,455 km and $g_0r_0{}^2$ = 400,000:

$$v_3 = \sqrt{800{,}000 \cdot \frac{6{,}455}{800{,}000 \cdot 806{,}455}} = 0.09 \text{ km/sec} = 90 \text{ m/sec.}$$

The tangential velocity may again be imparted by means of a directing shot with

$$\frac{\Delta m}{m} = \frac{0.09 - 0.00}{1.0} = 0.09,$$

i.e., a shell of say 1/11 of the present mass has to be fired at 1,000 m/sec at right angles with respect to the present direction of travel.

The velocity v_a close to the Earth (r_a) is then

$$v_a = 0.09 \cdot \frac{800{,}000}{6{,}455} = \sim 11.1 \text{ km/sec,}$$

thus approximately the same as calculated for a parabolic path before.

Since the measurement of velocity, to be done en route, as well as measurements of distance are possibly subject to error, a check during the travel and possibly a correction of the trajectory is desirable in the following way (see Figure 15):
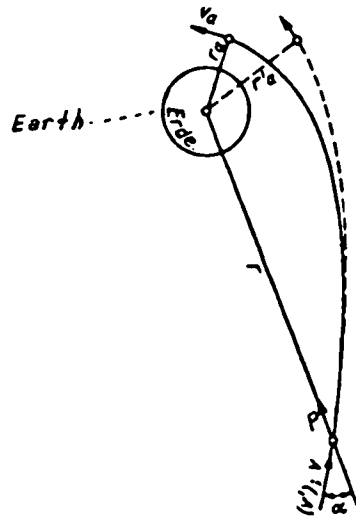
Figure 15.

At the distance r let us assume that subsequent measurements have established v' and the direction of the path (α) and that these lead to some undesirable perigee r_a'. If instead a perigee at the distance r_a is desired, then the following relations exist between r_a, r, α and the required velocities v_a and v (see the end of the last section):

1. By Gravitation

$$P = - g_o r_o^2 \frac{m}{r^2} ;$$

2. Conservation of Energy

$$\int P dr = - g_o r_o^2 m \int \frac{dr}{r^2} = \frac{mv^2}{2} - \frac{mv_a^2}{2}$$

or

$$+ \frac{g_o r_o^2}{r} + C = \frac{v^2}{2} - \frac{v_a^2}{2} ;$$

for $r = r_a$:

$$\frac{g_o r_o^2}{r_a} + C = 0;$$

thus

$$\frac{g_o r_o^2}{r} - \frac{g_o r_o^2}{r_a} = \frac{v^2}{2} - \frac{v_a^2}{2} ,$$

or

$$v_a^2 = v^2 + 2g_o r_o^2 \left(\frac{1}{r_a} - \frac{1}{r}\right);$$

3. According to the Area Law:

$$v \cdot r \cdot \sin \alpha = v_a \cdot r_a,$$

or

$$v_a^2 = \frac{v^2 r^2 \sin^2 \alpha}{r_a^2} ;$$

we must therefore have

$$v^2 \left(\frac{r^2}{r_a^2} \sin^2 \alpha - 1\right) = 2g_o r_o^2 \left(\frac{1}{r_a} - \frac{1}{r}\right) \tag{29}$$

or

$$v^2 = \frac{2g_o r_o^2}{r^2 \sin^2 \alpha - r_a^2} \cdot r_a \cdot \frac{r - r_a}{r} ,$$

and

$$v = \sqrt{\frac{2g_o r_o^2}{r^2 \sin^2 \alpha - r_a^2} r_a \frac{r - r_a}{r}} , \tag{30}$$

in place of v'.

If f.i. at the distance r_4 = 400,000 km the velocity v_4' = 1.415 km/sec and the direction α_4 = 7°50' are determined (both of which would correspond to a parabola with a perigee of r_a' = 7,500 km), then

$$\frac{r_4^2 \sin^2 \alpha_4}{r_a} = \frac{400,000^2 \cdot 0.137^2}{6,455} = 465,000 \text{ km}$$

and to achieve a perigee of r_a = 6,455 km, we must in accordance with (30)

$$v_4 = \sqrt{\frac{2g_o r_o^2}{r_4^2 \sin^2 \alpha_4 - r_a^2} \cdot r_a \frac{r_4 - r_a}{r_4}} = \sqrt{\frac{800,000}{465,000 - 6,455} \cdot \frac{400,000 - 6,455}{400,000}} =$$

$$1.31 \text{ km/sec,}$$

therefore

$$\Delta v_4 = v_4 - v_4' = 1.310 - 1.415 - 0.105 \text{ km/sec,}$$

i.e., the correction of the journey may again be done by a correcting shot with

$$\frac{\Delta m}{m} = \frac{\Delta v_4}{c} = \frac{0.105}{1.0} = 0.105$$

or approximately with 1/9.5 of the vehicle mass -- to be fired directly forward.

By means of (29) finally one can also take into account the influence of the Earth rotation, neglected so far. It imparts to the rising vehicle an initial velocity v_u at the equator $\frac{40,000 \text{ km}}{86,400 \text{ sec}} =$ 0.463 km/sec and at our latitude of approximately $50°$ about $0.463 \cdot \cos 50° = \sim 0.3$ km/sec. In consequence the vehicle does not rise in a straight trajectory, and the direction of motion at the distance r_1, after v_1 is reached, is not exactly radially, but inclined to the radius by an angle α_1, given by

$$\sin \alpha_1 = \frac{v_u}{v_1}$$

(see Figure 16).

With the previously found values of $r_1 = 8,490$ and $v_1 = 9.68$ km/sec, the further trajectory would now be a shallow parabola with a very close perigee of about 8 km. At $r_2 = 40,000$ km the velocity is

$$v_2' = \sqrt{\frac{2g_o r_o^2}{r_2}} = 4.46 \text{ km/sec}$$

and according to the area theorem

$$v_2 r_2 \sin \alpha_2 = v_1 r_1 \sin \alpha_1,$$

and therefore

$$\sin \alpha_2 = \sin \alpha_1 \cdot \frac{v_1 r_1}{v_2 r_2} = \frac{v_u r_1}{v_2 r_2} = \frac{0.3 \cdot 8,490}{4.46 \cdot 40,000} = 0.0143.$$
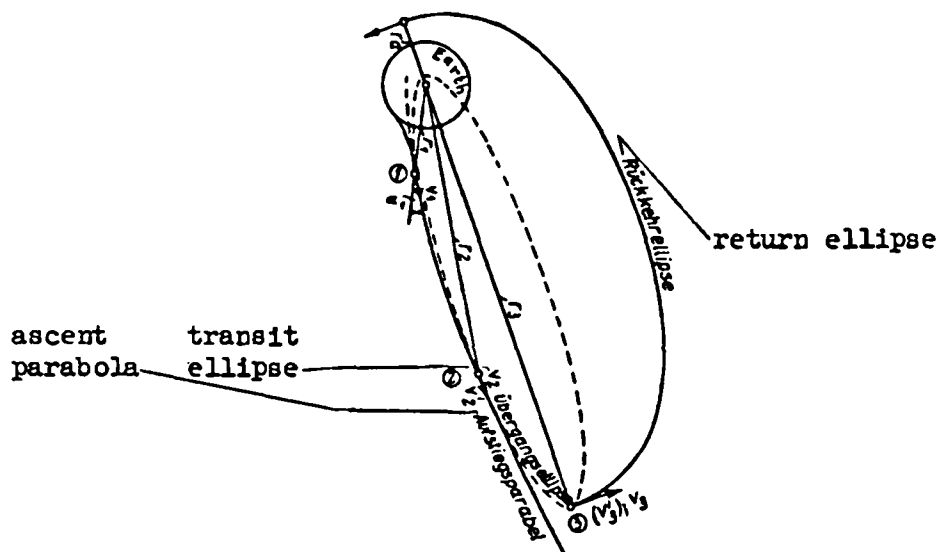
ascent
parabola

transit
ellipse

return ellipse

Figure 16.

If the orbit is again corrected by a shot of c = 1 km/sec and $\frac{\Delta m}{m}$ = 0.11 from v_2' = 4.46 to v_2 = 4.35 km/sec, there results a transition ellipse with maximum perigee and apogee given by (29):

$$\frac{v_2^2 r_2^2 \sin^2 \alpha_2}{r_3^2} - v_2^2 = \frac{2g_0 r_0^2}{r_3} - \frac{2g_0 r_0^2}{r_2} ;$$

$$r_3^2 \left(\frac{2g_0 r_0^2}{r_2} - v_2^2\right) - r_3 \cdot 2g_0 r_0^2 = - v_2^2 r_2^2 \sin^2 \alpha_2;$$

$$\substack{max \\ min} \ r_3 = \frac{g_0 r_0^2}{\frac{2g_0 r_0^2}{r_2} - v_2^2} \left[1 \pm \sqrt{1 - \left(\frac{v_2 r_2 \sin \alpha_2}{g_0 r_0^2}\right)^2 \left(\frac{g_0 r_0^2}{r_2} - v_2^2\right)}\right];$$

$$\substack{max \\ min} \ r_3 = \frac{400,000}{\frac{800,000}{40,000} - 4.35^2} \left[1 \pm \right.$$

$$\left. \sqrt{1 - \left(\frac{4.35 \cdot 40,000 \cdot 0.0143}{400,000}\right)^2 \left(\frac{800,000}{40,000} - 4.35^2\right)} \right]$$

$$\substack{max \\ min} \ r_3 = 370,500 \left[1 \pm 0.99999\right];$$

i.e., the perigee is only 4 km or nearly zero, and the apogee is about 741,000 km or nearly equal to the previous height of 800,000 km. On the other hand, at this distance r_3 = 741,000 km, the

velocity is not now zero, but, given by the area theorem

$$v_3 = \frac{v_2 r_2 \sin \alpha_2}{r_3} = \frac{4.35 \cdot 40,000 \cdot 0.0143}{741,000} = 0.0034 \text{ km/sec} = 3.4 \text{ m/sec},$$

tangentially.

To enter the desired return ellipse, we now need in place of the previous value $v_3 = 0.09$ km/sec according to (28)

$$v_3 = \sqrt{2g_o r_o^2 \frac{r_a}{r_3(r_3 + r_a)}} = \sqrt{800,000 \cdot \frac{6,455}{741,000 \cdot 747,455}} =$$

$$0.0964 \text{ km/sec} = 96.4 \text{ m/sec},$$

i.e.,

$$\Delta v = 96.4 - 3.4 = 93 \text{ m/sec},$$

so that

$$\frac{\Delta m}{m} = \frac{\Delta v}{c} = 0.093 = \frac{1}{10.8}$$

in place of the previous 1/11; the Earth rotation is therefore of no great significance.

An arbitrary course of the journey between launching and landing, according to what has been said above, presents no great difficulties.

If the velocity corrections, as assumed up to now, are done by means of individual shots, and if m_0 and m_1 denote masses before and after the shot, then by (1)

$$\frac{\Delta m}{m} = \frac{m_0 - m_1}{m_0} = \frac{\Delta v}{c}$$

or

$$\frac{m_0}{m_1} = \frac{1}{1 - \frac{\Delta v}{c}} \tag{31}$$

To avoid sudden shocks and to lower the weight of the cannon, it is desirable to replace the shots by several consecutive ones. In the limit this resembles the procedure of section I, i.e., mass radiation, then

$$\frac{dm}{m} = \frac{dv}{c}$$

or generally

$$\ln m = \frac{v}{c} + C.$$

If, at the beginning of the correction, the mass is m_0 and the velocity v_0, while at the end these are m_1 and v_1, then

$$\ln m_0 = \frac{v_0}{c} + C$$

$$\ln m_1 = \frac{v_1}{c} + C$$

consequently

$$\ln \frac{m_0}{m_1} = \frac{v_0 - v_1}{c} = \frac{\Delta v}{c}$$

and

$$\frac{m_0}{m_1} = e^{\frac{\Delta v}{c}} \tag{32}$$

Since only a mass decrease, never an increase, occurs, the sign of Δv refers to direction of the shot or radiation.

For smaller values of $\frac{\Delta v}{c}$ the results of (31) and (32) differ by only small amounts. For larger values however, radiation is more favorable than the individual shot; f.i.:

for $\frac{\Delta v}{c} = 0.1$ becomes $\frac{1}{1 - 0.1} = 1.11$ and $e^{0.1} = 1.105$

for $\frac{\Delta v}{c} = 0.5$ becomes $\frac{1}{1 - 0.5} = 2.0$ and $e^{0.5} = 1.65$

for $\frac{\Delta v}{c} = 0.9$ becomes $\frac{1}{1 - 0.9} = 10.0.$ and $e^{0.9} = 2.46$

for $\frac{\Delta v}{c} = 1.0$ becomes $\frac{1}{1 - 1} = \infty$ and $e^{1.0} = 2.72$

To determine the duration of free flight, let us ignore the small influence of the Earth rotation and let us assume further that $r_2 = r_1$. The flight then consists of two sections:

I. t_I from the end of the accelerating stage at r_1 = 8,490 km to the beginning of the return ellipse at r_3 = 800,000 km;

II. The time t_{II} to cover the return ellipse from the apogee at r_3 = 800,000 km to the largest perigee at r_a = 6,455 km.

The time t_I equals the free fall of a body with zero initial velocity from a height r_3 = 800,000 km down to r_1 = 8,490 km. Here at an arbitrary point r the velocity v according to (27) is:

$$v = \sqrt{2g_0 r_0^2 \frac{r_3 - r}{rr_3}}$$

or, since $v = -\dfrac{dr}{dt}$:

$$-\frac{dr}{dt} = \sqrt{\frac{2g_0 r_0^2}{r_3}} \cdot \sqrt{\frac{r_3 - r}{r}} \; ;$$

$$-\sqrt{\frac{2g_0 r_0^2}{r_3}} \cdot t = \int \frac{\sqrt{r}\; dr}{\sqrt{r_3 - r}} + C;$$

$$-\sqrt{\frac{2g_0 r_0^2}{r_3}} \cdot t = -\sqrt{r(r_3 - r)} + r_3 \arcsin \sqrt{\frac{r}{r_3}} + C;$$

for $r = r_3$:
$$0 = 0 + r_3 \cdot \frac{\pi}{2} + C;$$

thus generally:

$$\sqrt{\frac{2g_0 r_0^2}{r_3}} \cdot t = \sqrt{r(r_3 - r)} + r_3 \left(\frac{\pi}{2} - \arcsin \sqrt{\frac{r}{r_3}}\right),$$

and for $r = r_1$:

$$t_I = \sqrt{\frac{r_3}{2g_0 r_0^2}} \left[\sqrt{r_1(r_3 - r_1)} + r_3\left(\frac{\pi}{2} - \arcsin \sqrt{\frac{r_1}{r_3}}\right)\right];$$

for large values of r_3 with respect to r_1 -- is the case -- one may put

$$\arcsin \sqrt{\frac{r_1}{r_3}} = \sqrt{\frac{r_1}{r_3}}$$

so that

$$t_I = \sim \sqrt{\frac{r_3}{2g_0 r_0^2}} \left[\sqrt{r_1(r_3 - r_1)} + r_3\left(\frac{\pi}{2} - \sqrt{\frac{r_1}{r_3}}\right)\right];$$

also

$$t_I = \sqrt{\frac{800,000}{800,000}} \left[\sqrt{8,490\ (800,000 - 8,490)} + 800,000 \left(\frac{3.1416}{2} - \right.\right.$$

$$\left.\left. \sqrt{\frac{8,490}{800,000}}\right)\right] = 1 \cdot \left[81,900 + 1,174,400\right] = 1,256,300 \text{ sec} =$$

~ 349 hours.

The time t_{II} to cover half the elliptic orbit follows from (18a):

$$t_{II} = \frac{ab\pi}{v_3 r_3} ,$$

whereby

$$a = \frac{r_3 + r_a}{2} = \frac{800,000 + 6,455}{2} = 403,227 \text{ km}$$

and

$$b = \frac{v_3 r_3}{\sqrt{\frac{2 g_o r_o^2}{r_3} - v_3^2}} = \frac{0.09 \cdot 800,000}{\sqrt{\frac{800,000}{800,000} - 0.09^2}} = 72,400 \text{ km,}$$

therefore

$$t_{II} = \frac{403,227 \cdot 724,000 \cdot \pi}{0.09 \cdot 800,000} = 1,272,000 \text{ sec} = \sim 354 \text{ hours.}$$

The entire duration of free flight is therefore

$$t_I + t_{II} = 349 + 354 = 703 \text{ hours} = \sim 29\ 1/3 \text{ days}$$

and the entire trip, including launching and landing, takes

$$703 + 22.6 = 725.6 \text{ hours} = \sim 30\ 1/5 \text{ days,}$$

(roughly one month).

Our investigations so far permit a more accurate estimate of the vehicle weight G_1, so far assumed as 2 tons. It has to include:

a) the passengers and personal equipment,

b) liquid and solid food,

c) fuel for heating,

d) oxygen for breathing and combustion,

e) containers for the supplies mentioned,

f) equipment for heating, ventilation, waste removal, measurements and observations,

g) weight of the parachutes for the glide, consisting of breaking area, wing, elevation control, nose cone and air frame,

h) the weight of the fuselage wall,

i) the cannon and ammunition for the directional shots.

kg

For a) Two men with clothing and personal belongings to weigh at most 2 · 100 =   200

For b) The daily consumption per man of a suitable diet and water is about 4 kg; thus for 2 men/month: 2 · 30 · 4 =   240

For c) Since the vehicle radiates heat to space (no heat conduction), heat losses will not be larger than those of a Dewar of the same shape and dimension, i.e., very small for a smooth surface. If in addition the part of the shell facing the sun is partially or entirely painted black to take any more of the sun's heat, then the internal temperature will remain tolerable without further artifices. To calculate unfavorably, heat transfer is to be established for conduction and not radiation. The rate of heat loss/hour is then $V = \Delta t \cdot f \cdot \varphi$, where Δt represents the difference between internal and external temperature, f the separation area and φ the heat conduction/hour for $1m^2$ area/degree in heat units, which depends on the nature of the wall (1 heat unit = 1 k cal). By lagging the wall with a suitable insulator -- which must be as light as possible (say peatmoss) -- the heat conduction may be taken as $\varphi = 0.5$. The vehicle surface f is to be kept as small as possible; the sphere has max. volume for min. surface, but since for other reasons the min. dimension is to be 1.5 m and since the space (see Figure 13) for 2 persons and supplies should have a min. volume of 4.5 m^3, an ellipsoid of revolution, having

carry   440

1.6 m dia. and 3.4 m length with a volume of 4.55 m^3 and surface f = 14.45 m^2, may be used. The internal temperature may be + 10° C and it is further assumed that the wall, exposed to the sun, has a temperature of + 70° C, that facing space about - 270° C, i.e., the mean external temperature is - 100° C and the difference Δt = 110° C. The rate of heat loss is V = 110 · 14.45 · 0.5 = 800 k cal/hr, the daily loss 24 · 800 = 19,000 k cal. This loss must be compensated by burning a suitable fuel. The optimum fuel is petroleum with 11,000 k cal/kg, requiring a daily fuel consumption of at least $\frac{19,000}{11,000}$ = 1.7 kg and with regard to the requirements under d), we assume a fuel consumption of 2 kg/day, in 30 days therefore 30 · 2 =   60

For d) Since 1 kg of petroleum requires 2.7 kg of oxygen, a daily provision of 2 · 2.7 = 5.4 kg oxygen is required. Further, one man requires per day about 0.6 kg oxygen, thus 2 men 1.2 kg, making a total consumption of 5.4 + 1.2 = 6.6 kg daily and a grand total of 30 · 6.6 =   200

The oxygen is to be taken along in the liquid state and contained in vacuum containers, since the containers for compressed oxygen are too heavy, due to the wall thickness required to withstand the high pressures. Liquid oxygen however has a temperature of - 190° C; and for the transformation from the gaseous to the liquid state a latent heat of 500 k cal/kg is assumed. For the heating of the gaseous oxygen with a specific heat of 0.27 from - 190° to + 10° C, a further 0.27 · 200 = 54 k cal are necessary. This gives a total requirement of 6.6 · 554 = 3,560 k cal/day, to make available the necessary 6.6 kg of oxygen. This in turn requires $\frac{3,560}{11,000}$ = 0.3 kg of petroleum, thus increasing fuel consumption under c) from 1.7 kg by 0.3 kg to 2.0 kg, so that the quantity quoted under c) is enough.

For e) The vessels for storing liquid oxygen will be Dewar flasks with an assumed ratio of weight to that of the constants of 0.4, while the remaining supplies will be assumed contained in vessels with a

<div align="right">transfer   700</div>

weight ratio of 0.2, so that we get 200 · 0.4 +
(240 + 60) · 0.2 =   <div align="right">140</div>

For f) For an efficient petroleum stove, for
ventilation and garbage removal, for chronometers,
for protractors, distance meters and so on we assume   <div align="right">200</div>

For g) For the wings F_0 = 59 m^2 and breaking
area F = 6 m^2, elevation (and preferably also
lateral steering) control = 5 m^2. For the nose
cone, which, to decrease the weight and heat trans-
fer, is to be separated from the vehicle proper, a
cone of about 1.6 m base dia. and 4 m side; $1{,}6 \, \pi \cdot$
$\frac{4.0}{2}$ = 10 m^2 ; together 6 + 59 + 5 + 10 = 80 m^2 per
6 kg/m^2 =   <div align="right">240</div>

For h) The surface of the rump, according to c),
is 14.45 m^2, the weight, including heat insulation,
may be taken as 50 kg/m^2, a total of 14.45 · 50 =   <div align="right">780</div>

For i) The cannon   <div align="right">200</div>

Therefore the total weight without ammunition: <div align="right">2,260</div>

If the total loss of weight during the journey be-
cause of consumption of supply is ignored, and one as-
sumes three directing shots of 1/10 of the total mass
each, then there results, as the initial weight after
acceleration ceases, G_1 = 2,260 · 1.1^3 =   <div align="right">3,000</div>

thus the ammunition to be taken along is 3,000 - 2,260 =   740

At the beginning of the glide all supplies of ammunition, food,
fuel and oxygen are used up and the remaining weight is

G_1' = 3,000 - 740 - 240 - 60 - 200 = 3,000 - 1,240 = 1,760 kg.

The resulting final weight at landing is therefore a little
less than the 2 tons assumed in section II, on the other hand, the
initial weight is about 1.5 of that given in section I. Therefore,
according to section I, the mass radiated during acceleration, has
to increase also by the factor of 1.5, i.e., the length dimensions
of Figure 4 would increase, other things remaining equal by
$(1.5)^{1/3}$. If at the same time air resistance on the way up is
considered also, which according to the end of section I would

require an increase in initial mass m_0 in the ratio $\frac{933}{825}$ then the required linear increase, according to Figure 4, is given by

$$\sqrt[3]{1.5 \cdot \frac{933}{825}} = \sqrt[3]{1.69} = 1.192,$$

so that for $c = 2,000$ m/sec and $\alpha c = 30$ m/sec^2

the height $\qquad$ 27 $\cdot$ 1.192 $= \sim 32$ m

the lower diameter $\qquad$ 18.7 $\cdot$ 1.192 $= \sim 22$ m

the upper diameter $\qquad$ 0.65 $\cdot$ 1.192 $= \sim 0.77$ m

and the weight at the beginning of the ascent will be

$$G_0 = G_1 \cdot \frac{m_0}{m_1} = 3 \cdot 933 = 2,799 \text{ tons.}$$

The use of a single gun presupposes that the vehicle can be rotated arbitrarily, which is possible, if a part of the mass in the vehicle is turned in the opposite direction, f.i. by having the passengers climb about the wall of the vehicle by means of rungs, put there for this purpose. If the living masses m_1 here move with an angular velocity ω_1 at an average distance x_1 from the vehicle center of mass, while the dead weight m_t moves with an opposed angular velocity ω_t at a distance x_t from the center of mass, then, since the total angular momentum must remain zero,

$$\Sigma mvx = 0, \text{ or, da } v = x \cdot \omega,$$

$$\Sigma m\omega x^2 = 0$$

or

$$m_t \cdot \omega_t \cdot x_t^2 = m_1 \cdot \omega_1 \cdot x_1^2;$$

therefore

$$\frac{\omega_t}{\omega_1} = \frac{m_1 \cdot x_1^2}{m_t \cdot x_t^2}, \tag{33}$$

i.e., the angular velocities are inversely as the moments of inertia of the masses. If a total weight of 140 kg is assumed for the passengers, leaving a dead weight for the vehicle in the most unfavorable case (i.e., at the beginning of free travel) 3,000 - 140 = 2,860 kg, then there follows with the average distances to the center of gravity, given by Figure 17:
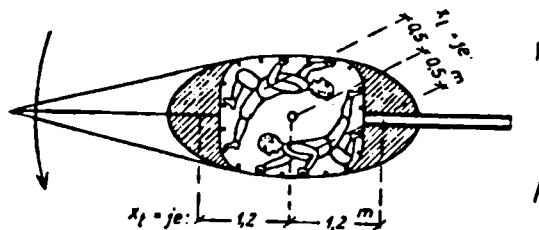
Figure 17.

$$\frac{\omega_t}{\omega_1} = \frac{140 \cdot 0.5^2}{2,860 \cdot 1.2^2} = \sim \frac{1}{120} \; .$$

Thus, to effect a 360° rotation, the inmates have to climb 120 times around the wall, for 180° 60 times and 30 times for 90°. Since this will give them the illusion of gravity under hands and feet, this climbing exercise will be a welcome change in the otherwise gravitation-less existence. If they move their mass centers with a velocity of 0.5 m/sec, then they will need approximately $\frac{1.0\pi}{0.5} = 6$ sec for a 90° rotation, i.e., 30 · 6 = 180 sec. Since at a distance r_2 = 40,000 km from the center of the Earth, where the first shot is to be fired, the trajectory velocity is about 4.46 km/sec, a distance of 4.46 · 180 = 800 km is covered, before the laterally positioned vehicle is brought into the position required to change the velocity by Δv_2 ($\pm$ corresponding to the sign of v_2 with respect to the gun position). With respect to the 40,000 km, this difference of 800 km is insignificant.

The correct positioning for the wings at the beginning of the glide, a rotation about the main axis of the ellipsoid may be similarly effected, but will be a little faster, since the dead weight has less distance from the main axis.

* * *

At the end of this section, we want to derive in brief the laws of motion in a gravitational field, which have been and will be used repeatedly.

1. Experimental fact: The planets describe nearly circular orbits.

2. If a body, mass m, describes a circular orbit r with velocity v, then the centripetal acceleration $\frac{dv_r}{dt}$, according to

Figure 18, is given as follows:
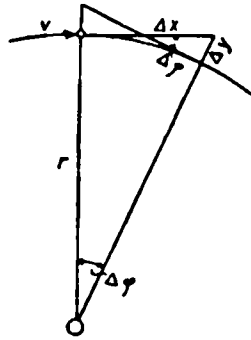
Figure 18.

After a short time Δt, the path covered is given by

$$\Delta x = v \cdot \Delta t \text{ or } \Delta t = \frac{\Delta x}{v},$$

and

$$\Delta y = \frac{dv_r}{dt} \cdot \frac{(\Delta t)^2}{2} = \frac{dv_r}{dt} \cdot \frac{(\Delta x)^2}{2v^2};$$

and because of the similarity of the right-angled triangles with angle $\Delta \varphi$:

$$\Delta y = \frac{\Delta x}{2} \cdot \frac{\Delta x}{r} = \frac{(\Delta x)^2}{2r}.$$

Comparing the two expressions for Δy

$$\frac{dv_r}{dt} = \frac{v^2}{r},$$

or, if a central force P produces this centripetal acceleration:

$$P = - m \cdot \frac{v^2}{r} \tag{34}$$

(the - sign, because P is opposed to r).

3. Experimental fact: The squares of the periods T_1 and T_2 of two planets are proportional to the cubes of their distances r_1 and r_2 (Figure 19) from the sun, or

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$
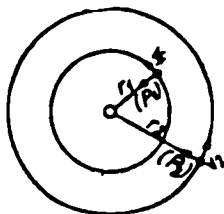
Figure 19.

If v_1 and v_2 are the respective velocities, then

$$T_1 = \frac{2r_1\pi}{v_1} \text{ and } T_2 = \frac{2r_2\pi}{v_2},$$

thus

$$\frac{r_1^2}{v_1^2} \cdot \frac{v_2^2}{r_1^2} = \frac{r_1^3}{r_2^3}$$

or

$$\frac{v_2^2}{v_1^2} = \frac{r_1}{r_2}. \tag{35}$$

4. From equations (34) and (35):

$$\frac{P_1}{P_2} = \frac{\dfrac{m_1 v_1^2}{r_1}}{\dfrac{m_2 v_2^2}{r_2}} = \frac{m_1 v_1^2 r_2}{m_2 v_2^2 r_1} = \frac{m_1 r_2^2}{m_2 r_1^2}$$

and therefore

$$\left.\begin{array}{l} P_1 = -\mu \cdot \dfrac{m_1}{r_1^2} \\[2em] P_2 = -\mu \cdot \dfrac{m_2}{r_2^2} \end{array}\right\} \quad \begin{array}{l}\text{(negative, because P is}\\ \text{directed toward and r}\\ \text{away from the center);}\end{array}$$

or generally the law:

$$P = -\mu \cdot \frac{m}{r^2}, \tag{36}$$

where μ is a ratio depending on the center of attraction.

5. For the sun as a center, μ follows from the Earth's mean distance r_e = 149,000,000 km, moving with a period of T_e = 365 days, i.e., with a mean velocity of

$$v_e = \frac{2 r_e \pi}{T_e} = \frac{2 \cdot 149,000,000 \cdot \pi}{365 \cdot 86,400} = 29.7 \text{ km/sec},$$

so that by (34) and (36):

$$- P = m_e \cdot \frac{v_e^2}{r_e} = \mu \cdot \frac{m_e}{r_e^2} ,$$

or

$$\mu = v_e^2 \cdot r_e = (27.7 \text{ km/sec})^2 \cdot 149,000,000 \text{ km},$$

$$\mu = 132,000,000,000 \frac{\text{km}^3}{\text{sec}^2} . \tag{37}$$

6. For the Earth as center, μ follows from the distance r_m = 392,000 km and period of 28 days, i.e., with a mean velocity of

$$v_m = \frac{2 r_m \pi}{T_m} = \frac{2 \cdot 392,000 \pi}{28 \cdot 86,400} = 1.01 \text{ km/sec},$$

of the moon, so that

$$\mu = v_m^2 \cdot r_m = 1.01^2 \cdot 392,000 = 400,000 \frac{\text{km}^3}{\text{sec}^2} . \tag{38}$$

7. At the Earth surface (r_o = 6,380 km) the central force by (36) should be:

$$P_o = \frac{\mu \cdot m}{r_o^2} = \frac{400,000}{6,380^2} \cdot m;$$

or in acceleration

$$g_o = \frac{\mu}{r_o^2} = \frac{400,000}{6,380^2} = 0.0098 \text{ km/sec}^2 = 9.8 \text{ m/sec}^2;$$

which would follow immediately, from observations of free fall on Earth, giving

$$\mu = g_o r_o^2 = 0.0098 \cdot 6,380^2 = 400,000 \frac{\text{km}^2}{\text{sec}^2} .$$

8. The area theorem: For central motion of a mass point under the influence of a force P, directed toward a fixed center, the following is true:

At the distance r_1 the velocity v_1 changes magnitude and direction in consequence of P_1, which causes central acceleration. The new velocity v_2 may be regarded as the diagonal of a parallelogram of velocities. The area covered by the radius vector is in unit time according to Figure 20:
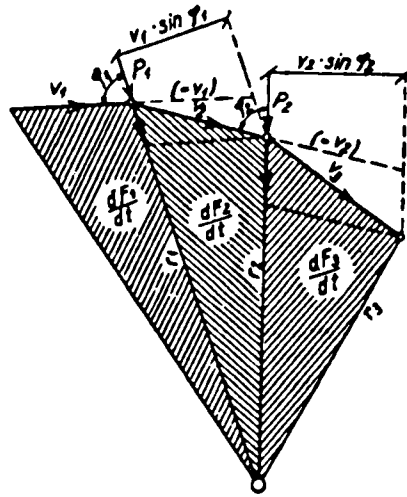

Figure 20.

for a velocity v_1:

$$\frac{dF_1}{dt} = \frac{r_1 \cdot v_1 \sin \varphi_1}{2},$$

for a velocity v_2:

$$\frac{dF_2}{dt} = \frac{r_1 \cdot v_1 \sin \varphi_1}{2}.$$

Similarly the velocity v_3 at r_2, and given by v_2 and P_2, is to be regarded as a diagonal in such a parallelogram. The rate, at which the radius vector covers the area, is then:

for a velocity v_2:

$$\frac{dF_2}{dt} = \frac{r_2 \cdot v_2 \sin \varphi_2}{2},$$

for a velocity v_3:

$$\frac{dF_3}{dt} = \frac{r_2 \cdot v_2 \sin \varphi_2}{2}.$$

This gives

$$\frac{dF_1}{dt} = \frac{dF_2}{dt} = \frac{dF_3}{dt} = \text{constant} \tag{39}$$

i.e., equal areas are swept in equal times.

9. Energy conservation: At each point, according to Figure 1, the force consists of two components with fixed directions X and Y, so that:

$$X = m \cdot \frac{dv_x}{dt} \quad \text{and} \quad Y = m \cdot \frac{dv_y}{dt} ;$$

where

$$\frac{dx}{dt} = v_x \quad \text{and} \quad \frac{dy}{dt} = v_y;$$

from this $\overline{X \cdot dx = mv_x dv_x}$ and $\overline{Y \cdot dy = mv_y dv_y};$

$$\int Xdx = \frac{mv_x^2}{2} - \frac{mv_{ax}^2}{2} ; \quad \int Ydy = \frac{mv_y^2}{2} - \frac{mv_{ay}^2}{2} ;$$

or, since

$$v^2 = v_x^2 + v_y^2,$$

between two points with velocities v_a and v:

$$\int Xdx + \int Ydy = \frac{mv^2}{2} - \frac{mv_a^2}{2} \cdot {}_{\text{μ}}$$

Further by Figure 21:
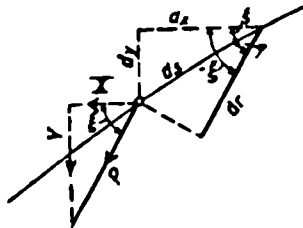
Figure 21.

$$X = P \cdot \cos \xi; \qquad dx = ds \cdot \cos \zeta; \qquad ds = \frac{dr}{\cos \varphi} ;$$

$$Y = P \cdot \sin \xi; \qquad dy = ds \cdot \sin \zeta;$$

and

$$\int P(\cos \xi \cos \zeta + \sin \xi \sin \zeta) \frac{dr}{\cos \varphi} = \frac{mv^2}{2} - \frac{mv_a^2}{2},$$

or, since $\cos \xi \cos \zeta + \sin \xi \sin \zeta = \cos (\xi - \zeta) = \cos \varphi$:

$$\int Pdr = \frac{mv^2}{2} - \frac{mv_a^2}{2}. \qquad (40)$$

10. Application to arbitrary motion under gravity: In Figure 22, Z is the center of attraction, v_a the orbital velocity of a body at its perigee r_a, v the orbital velocity at an arbitrary distance r with components along and at right angles to the radius vector, given by $\frac{dr}{dt}$ and $r \cdot \frac{d\varphi}{dt}$ respectively, we have by the law of equation (36):

$$P = - \frac{\mu \cdot m}{r^2};$$

according to theorem equation (40):

$$\int Pdr = - \mu m \int \frac{dr}{r^2} = \frac{mv^2}{2} - \frac{mv_a^2}{2},$$

or

$$+ \frac{\mu}{r} + C = \frac{v^2}{2} - \frac{v_a^2}{2};$$

for $r = r_a$:

$$\frac{\mu}{r_a} + C = 0$$

thus

$$\frac{\mu}{r} - \frac{\mu}{r_a} = \frac{v^2}{2} - \frac{v_a^2}{2}$$

or

$$v^2 = v_a^2 + \frac{2\mu}{r} - \frac{2\mu}{r_a}; \qquad (41)$$
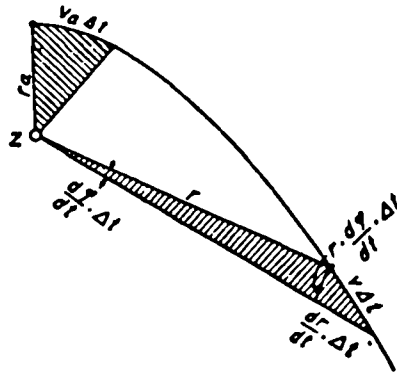
Figure 22.

by the theorem (39):

$$\frac{v_a \cdot \Delta t \cdot r_a}{2} = (r + \frac{dr}{dt} \cdot \Delta t) \cdot \frac{r}{2} \cdot \frac{d\varphi}{dt} \cdot \Delta t;$$

from this

$$\frac{d\varphi}{dt} = \frac{v_a \cdot r_a}{r^2 + r \frac{dr}{dt} \cdot \Delta t};$$

or for $\Delta t = dt = 0$:

$$\frac{d\varphi}{dt} = \frac{v_a r_a}{r^2};$$

(42)

by Pythagoras:

$$(v\Delta t)^2 = (\frac{dr}{dt} \cdot \Delta t)^2 + (r \frac{d}{dt} \cdot \Delta t)^2$$

or

$$v^2 = (\frac{dr}{dt})^2 + r^2(\frac{d\varphi}{dt})^2 = (\frac{dr}{dt})^2 + \frac{v_a^2 r_a^2}{r^2};$$

by comparison with (41):

$$(\frac{dr}{dt})^2 = v_a^2 + \frac{2\mu}{r} - \frac{2\mu}{r_a} - \frac{v_a^2 r_a^2}{r^2};$$

further by (42):

$$(\frac{d\varphi}{dt})^2 = \frac{v_a^2 r_a^2}{r^4};$$

therefore

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{v_a^2 r_a^2}\left(v_a^2 - \frac{2\mu}{r_a} + \frac{2\mu}{r} - \frac{v_a^2 r_a^2}{r^2}\right)$$

or

$$\frac{dr}{d\varphi} = r\sqrt{\frac{v_a^2 - \frac{2\mu}{r_a}}{v_a^2 r_a^2} r^2 + \frac{2\mu}{v_a^2 r_a^2} r - 1} .$$ (43)

11. The equation of an ellipse (see Figure 23):
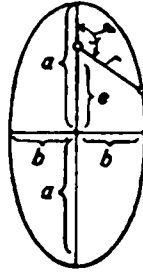
Figure 23.

$r = \dfrac{b^2}{a + e \cos \varphi}$, where $e^2 = a^2 - b^2$ or $a^2 - e^2 = b^2$.

$$\frac{dr}{d\varphi} = \frac{b^2 \cdot e \cdot \sin \varphi}{(a + e \cos \varphi)^2} ;$$

here we may put

$$\frac{b^2}{(a + e \cos \varphi)^2} = \frac{r^2}{b^2}$$

and

$$e \sin \varphi = \sqrt{e^2 - e^2 \cos^2 \varphi} ,$$

further

$$e^2 \cos^2 \varphi = \left(\frac{b^2}{r} - a\right)^2 = \frac{b^4}{r^2} - \frac{2ab^2}{r} + a^2 ,$$

therefore

$$e \sin \varphi = \sqrt{e^2 - a^2 + \frac{2ab^2}{r} - \frac{b^4}{r^2}} = \sqrt{- b^2 + \frac{2ab^2}{r} - \frac{b^4}{r^2}} \; ;$$

we have

$$\frac{dr}{d\varphi} = \frac{r^2}{b^2} \sqrt{- b^2 + \frac{2ab^2}{r} - \frac{b^4}{r^2}}$$

or

$$\frac{dr}{d\varphi} = r \sqrt{- \frac{1}{b^2} r^2 + \frac{2a}{b^2} r - 1} \; . \tag{44}$$

12. By comparison (43) and (44) for $\frac{dr}{d\varphi}$ it follows that the motion of a body under gravity (36) is an ellipse, for which

$$- \frac{1}{b^2} = \frac{v_a^2 - \frac{2\mu}{r_a}}{v_a^2 r_a^2} \; ,$$

and

$$\frac{2a}{b^2} = \frac{2\mu}{v_a^2 r_a^2} \; ;$$

therefore

$$a = \frac{\mu}{\frac{2\mu}{r_a} - v_a^2} \; ; \tag{45}$$

further

$$b^2 = a \, \frac{v_a^2 r_a^2}{\mu} = \frac{v_a^2 r_a^2}{\frac{2\mu}{r_a} - v_a^2} \; ,$$

thus

$$b = v_a r_a \sqrt{\frac{a}{\mu}} = \frac{v_a r_a}{\sqrt{\frac{2\mu}{r_a} - v_a^2}} \; ; \tag{46}$$

further

$$e^2 = a^2 - b^2 = a^2 - a \, \frac{v_a^2 r_a^2}{\mu} \; ;$$

by adding

$$0 = + 2ar_a - 2ar_a$$

there follows

$$e^2 = a^2 - 2ar_a + \frac{r_a^2 \cdot a}{\mu} \left(\frac{2\mu}{r_a} - v_a^2 \right),$$

or, since

$$\frac{1}{\mu} \cdot \left(\frac{2\mu}{r_a} - v_a^2 \right) = \frac{1}{a} \text{ is:}$$

$$e^2 = a^2 - 2ar_a + r_a^2 = (a - r_a)^2;$$

thus

$$e = \pm \, (a - r_a);$$

i.e., the center of attraction Z is the focus of the ellipse (see Figure 22 and 23).

13. As long as $\frac{2\mu}{r_a} - v_a^2 > 0$, a remains positive, b real, i.e., the orbit is an ellipse.

If $\frac{2\mu}{r_a} - v_a^2 = 0$, then a is infinite, b infinite, the orbit is now a parabola.

If $\frac{2}{r_a} - v_a^2 < 0$, then a is negative, b imaginary and the orbit a hyperbola.

If a is to be equal to r_a, we must have

$$r_a = \frac{\mu r_a}{\frac{2\mu}{r_a} - v_a^2} \, ,$$

or

$$2\mu - v_a^2 r_a = \mu;$$

thus

$$v_a^2 = \frac{\mu}{r_a} \; ;$$

the orbit in this case is a circle.

14. The period for an elliptic orbit is given by the area theorem (39):

$$\frac{dF}{dt} = \text{constant} = \frac{v_a r_a}{2} \; :$$

$$F = \frac{v_a r_a}{2} \cdot t = ab\pi \; ;$$

thus

$$t = \frac{2ab\pi}{v_a r_a} \; ; \qquad\qquad (47)$$

and if, according to (46), we substitute the value

$$b = v_a r_a \sqrt{\frac{a}{\mu}}$$

then we get:

$$t = 2a\pi \sqrt{\frac{a}{\mu}} = 2\pi \sqrt{\frac{a^3}{\mu}} \; . \qquad\qquad (48)$$

IV

CIRCUMNAVIGATION OF OTHER HEAVENLY BODIES

A circumnavigation of the Moon, f.i., to find the nature of its unknown side, will not substantially differ from free space travel, as long as one does not approach it close enough, so that its attraction as well as that of the Earth (which at the same distance is 80 times as effective) becomes significant. Since during the 30 days of the journey the Moon will also orbit the Earth once, we cannot speak of a circumnavigation, but rather a crossing of paths, which may take the form shown in Figure 24, where E, M and F are Earth, Moon and rocket respectively, while the numbers indicate simultaneous positions. The largest Moon perigee is therefore about half of the largest Earth apogee, the relative largest attraction by the Moon therefore about $\frac{4}{80} = \frac{1}{20}$ of the simultaneous Earth attraction. Its influence will not further be investigated here.
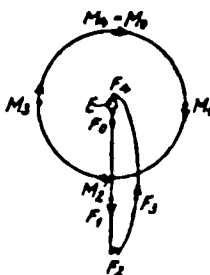
Figure 24.

In our consideration only the Earth attraction was considered, while that of the Sun was ignored, because the vehicle follows the 30 km/sec, which the Earth covers in its orbit about the Sun. This is only strictly true at the instance, when the vehicle is at rest relative to Earth, i.e., at its highest point r_3, and even then only if this point has the same distance from the Sun as the Earth. Assuming that the vehicle leaves the Earth tangentially to the Earth orbit, then its velocity, if 10 km/sec relative to Earth is with respect to the Sun 30 + 10 = 40 or 30 - 10 = 20 km/sec, according to a positive or negative direction. In the latter case, its instantaneous trajectory has a higher curvature, in the first case, less curvature than the Earth orbit, because of the solar attraction. Since the vehicle's velocity relative to Earth however quickly decreases because of the Earth's gravitational effect, and the total time of the ascent is only 15 days, i.e., 1/24 of an Earth orbit, the trajectory in the range considered will not perceptably deviate from the Earth orbit. If on the other hand the ascent is radial with respect to the Earth orbit, then at the apogee r_3 the velocity of the vehicle relative to the Sun equals that of the Earth, but the distance from the Sun is larger or smaller than the Earth-Sun distance, depending on the ascent being from or away from the Sun. In the former case, the trajectory again has a higher curvature, in the latter a lower curvature. But since an apogee of 800,000 km is negligible compared to the distance of 150,000,000 km, the deviation here too is hardly noticeable. The direction of ascent is therefore so far arbitrary. It is advisable however to direct it toward the Sun, so that the Earth can be seen in its entirety and brightly, which is necessary for the distance and velocity measurements. The distance r_3 = 800,000 km to be attained will therefore always be assumed in that direction during our further considerations, even if r_3 is ignored with respect to the distance from the Sun.

If at that point the tangential velocity v_3 is chosen 3 km/sec instead of 0.09 km/sec (see Figure 14) as in section III, then under the influence of the Earth's attraction alone the trajectory is now a flat hyperbola rather than an ellipse, since

$$\frac{2\mu}{r_3} - v_3{}^2 = \frac{2 \cdot 400,000}{800,000} - 3^2 = -8$$

and the vehicle will pursue a path directed away from the effective range of the Earth gravitation with nearly uniform velocity, until it finally -- so to say as an independent comet -- it is subject only to the attraction of the Sun. Initially the tangential velocity relative to the Sun is $v_I = 29.7 \pm 3.0 = 32.7$ or 26.7 km/sec, according to whether v_3 is along or against the Earth velocity of 29.7 km/sec. In either case the vehicle describes an ellipse about the Sun in the first case outside, in the second inside of the Earth orbit.

If the vehicle's path is to touch at a distance r_{II} from the Sun, the orbit of a planet other than the Earth, distant r_I from the Sun (see Figure 25), then the major axis of the ellipse is

$$a = \frac{r_I + r_{II}}{2} ;$$

and by (45)

$$a = \frac{\mu}{\frac{2\mu}{r_1} - v_I{}^2} ;$$

thus

$$\frac{2\mu}{r_I} - v_I{}^2 = \frac{2\mu}{r_I + r_{II}} ;$$

from this

$$v_I{}^2 = \frac{2\mu}{r_I + r_{II}} \cdot \frac{r_{II}}{r_I} ;$$

or

$$v_I = \sqrt{\frac{2\mu}{r_I + r_{II}} \cdot \frac{r_{II}}{r_I}} . \tag{49}$$

The Earth's mean distance from the Sun is $r_I = 149,000,000$ km, that of Venus f.i. $r_{II} = 108,000,000$ km. Since further by (37) $\mu = 132,000,000,000$ km^3/sec^2, we have for a journey close to Venus

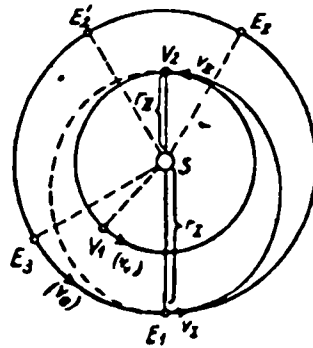$$v_I = \sqrt{\frac{264,000}{257} \cdot \frac{108}{149}} = 27.3 \text{ km/sec}.$$

Figure 25.

Now the Earth velocity is $v_e = 29.7$ km/sec and accordingly the velocity to be imparted to the vehicle, after it reaches its apogee, has to be

$$\Delta v_I = v_I - v_e = 27.3 - 29.7 = -2.4 \text{ km/sec}$$

and could be the result of a tangential shot of mass

$$\Delta m = m \cdot \frac{\Delta v_I}{c} ,$$

where m is the mass of the vehicle before the shot and c is the velocity of the projectile. In this case, one can no longer use the value of $c = 1$ km/sec, assumed in section III, and also a single shot of the required strength would endanger the vehicle and its passengers. A system of continuous mass radiation as in section I must be used with a min. velocity of $c = 2$ km/sec. We have now for the ratio of total mass before and after the action by (32)

$$\frac{m_o}{m_1} = e^{\left(\frac{\Delta v}{c}\right)} .$$

But since during the initial parallel paths of planet and vehicle orbit interference is unavoidable, an additional safety factor /see Note/, say $\nu = 1.1$, must be added, which necessitates:

$$\left(\frac{m_o}{m_1}\right)_I = \nu \cdot e^{\frac{\Delta v_I}{c}} = 1.1 \cdot e^{\frac{2.4}{2.0}} = 1.1 \cdot e^{1.20} = 3.65,$$

where the mass has to be thrown forward in the direction of the Earth's motion.

_____

/Note/ Such interferences may be obviated by radiating the mass $\frac{dm}{dt} = -am$ (see (1c)), directed against the disturbing planet and

equal to the gravitational effect g, so that at a distance x from the planet by (1a) and (2)

$$\frac{dv}{dt} = c\alpha = g = g_o \frac{r_o{}^2}{x^2} \text{ and } \frac{m_o}{m} = e^{\alpha t}.$$

At the assumed initial point x = 800,000 km from the Earth with g_o = 9.8 m/sec^2 and r_o = 6,380 km:

$$c\alpha = 9.8 \cdot \frac{6,380^2}{800,000^2} = \frac{1}{16,000} \text{ m/sec}^2$$

and after one day = 86,400 sec, when c = 2,000 m/sec:

$$\alpha t = \frac{c\alpha}{c} \cdot t = \frac{86,400}{16,000 \cdot 2,000} = 0.0270;$$

at the distance x = 800,000 km from Venus with g_o = 8.7 and r_o = 6,090;

$$c\alpha = 8.7 \cdot \frac{6,090^2}{800,000^2} = \frac{1}{20,000} \text{ m/sec}^2$$

and

$$\alpha t = \frac{86,400}{20,000 \cdot 2,000} = 0.0216;$$

at a distance x = 800,000 km from Mars with g_o = 3.7 and r_o = 3,392:

$$c\alpha = 3.7 \cdot \frac{3,393^2}{800,000^2} = \frac{1}{150,000}$$

and

$$\alpha t = \frac{86,400}{150,000 \cdot 2,000} = 0.00288.$$

With each succeeding day, x becomes larger, i.e., the daily increase αt smaller. By plotting planet and vehicle positions, one gets for the first 5 days the following distances x and the corresponding daily values αt:

Days	Earth		Venus		Mars	
	x km	αt	x km	αt	x km	αt
0	800,000	0.0270	800,000	0.0216	800,000	0.0029
1'	850,000	0.0240	850,000	0.0191	900,000	0.0023

Days	Earth x km	αt	Venus x km	αt	Mars x km	αt
2	900,000	0.0213	900,000	0.0170	1,000,000	0.0018
3	1,000,000	0.0173	1,000,000	0.0138	1,200,000	0.0013
4	1,100,000	0.0143	1,200,000	0.0096	1,400,000	0.0009
5	1,200,000	0.0120	1,400,000	0.0070	1,700,000	0.0006
Sum	$\sum \alpha t = 0.1159$		$\sum \alpha t = 0.0881$		$\sum \alpha t = 0.0098$	

Accordingly after the first 5 days $\nu = \dfrac{m_o}{m} = e^{\sum \alpha t}$:

for Earth: $\nu = e^{0.116} = 1.123$; for Venus: $\nu = e^{0.088} = 1.093$;

for Mars: $\nu = e^{0.01} = 1.01$.

The above safety factor $\nu = 1.1$ is thus only a rough mean, which must be corrected for each planet. The interference correction need not be done in one step, it will be sufficient to do it daily once or several times with corresponding intensity. /End of note/) The time required to cover half the ellipse by (48), using $a = \dfrac{r_I + r_{II}}{2} = 128,500,000$ km:

$$T_I = \pi \sqrt{\frac{a^3}{\mu}} = \pi \sqrt{\frac{128,500,000^3}{132,000,000,000}} = 12,600,000 \text{ sec} = 146 \text{ days}.$$

The Earth moves in its orbit with an angular velocity of $\dfrac{360^o}{365 \text{ days}} = 0.987^o/\text{day}$, Venus with $\dfrac{360^o}{224 \text{ days}} = 1.607^o/\text{day}$. During the time of 146 days, the Earth thus covers an arc of $146 \cdot 0.987 = 144^o$, Venus an arc of $146 \cdot 1.607 = 234.5^o$. In order to have the vehicle approach Venus in fact (say at a distance of 800,000 km from the center of Venus and on the side closest to the Sun), the launching has to take place at a time when Venus is $234.5 - 180 = 55.5^o$ behind Earth in the sense of the motion of the planets (points V_1 and E_1 in Figure 25). If the vehicle would continue its journey unchanged, then it would return after a further 146 days to its initial point in space via the dotted half of the ellipse, the Earth however would be retarded with respect to the vehicle by a further 36° or a total of 72° (point E_3 in Figure 25). To make it possible for the orbits to intersect, the time for the return trip must somehow be increased. Two possibilities present themselves:

1st possibility (see Figure 25). If the dotted branch of the ellipse is to lead back to Earth, then the Earth at the time of

departure at V_2 would have to be 36° in front of, rather than be-
hind, Venus, i.e., at E_2', not E_2. The vehicle would have to be
kept near Venus until the correct position of the two planets oc-
curs, i.e., until Venus has almost caught up again with Earth,
except for the 36°. Because of its faster travel, Venus gains
a daily angle of $1.607 - 0.987 = 0.62^O$ and so it would require
464 Earth days for it to advance from its initial 36° advantage
over Earth, the remaining 288° to arrive 36° behind the Earth.
During this time, the vehicle can be made to circle Venus arbi-
trarily often. To achieve this, it must first of all be suitably
decelerated, say by Δv_{II} and thus subjected to the permanent in-
fluence of the gravitational field of that planet, just as it was
previously taken out of the Earth's gravitational field by Δv_I.
The Venus near point V_2 (Figure 25) is attained with a velocity

$$v_{II} = v_I \cdot \frac{r_I}{r_{II}} = 27.3 \cdot \frac{149}{108} = 37.6 \text{ km/sec,}$$

while the orbital velocity of Venus is

$$v_V = \frac{2 \cdot 108,000,000 \cdot \pi}{224 \cdot 86,400} = 35.1 \text{ km/sec.}$$

To attain velocity zero with respect to Venus, a decrease of
$37.6 - 35.1 = 2.5$ km/sec must be made. If the orbit about Venus
is to be a circle of radius a, then the period by (48) is t =
$2\pi\sqrt{\frac{a^3}{\mu}}$. With regard to the correct vehicle position with respect
to the later departure, the following has to be observed concern-
ing the choice of t: During the 464 Earth days, Venus covers its
orbit $\frac{464}{224} = 2.07 = 2 + 0.07$ times , i.e., when the orbitting has
to stop, Venus will be further in its orbit by 0.07 rotations
about the Sun than at the beginning (see Figure 25a). Since the
vehicle's velocity, when entering the field of attraction of Venus
(v_{II}), as well as at the exit from this field (v_{II}'), must be di-
rected at right angles to the Sun-Venus radius vector, there are,
according to Figure 25a, at the moment, when the vehicle leaves
the orbit, 0.07 parts of an orbit missing. The total number of
orbits may therefore be 3.93 or 4.93 or 5.93 and so on, so that
f.i. for 5.93:

$$t = \frac{464}{5.93} = 78.2 \text{ days} = 6,750,000 \text{ sec.}$$

If mass conditions pertaining to Earth are assumed to apply,
for simplicity's sake, also in the case of Venus, which is of the
same size or nearly so (exact observations of the trajectory devia-
tions of comets indicate, that Venus has a mass only of 0.82 that

of Earth), then we may again put μ = 400,000 km^3/sec^2 and this leads for a to:

$$a = \sqrt[3]{\mu(\tfrac{t}{2\pi})^2} = \sqrt[3]{400,000 \ (\tfrac{6,750,000}{2\pi})^2} = 773,000 \text{ km},$$

and for a trajectory velocities during the orbits of

$$v_3 = \frac{2a\pi}{t} = \frac{2 \cdot 773,000 \cdot \pi}{6,750,000} = 0.72 \text{ km/sec.}$$

The desired orbit will automatically occur, if at the time of transit at V_2 (Figure 22) the relative velocity is not zero, but 0.72 km/sec, i.e., the retardation not equal to 2.5, but

$$\Delta v_{II} = 37.6 - 35.1 - 0.72 = \sim 1.8 \text{ km/sec.}$$

This again requires a radiation of mass of

$$(\frac{m_o}{m_1})_{II} = \nu \cdot e^{(\frac{\Delta v_{II}}{c})} = 1.1 \cdot e^{\frac{1.8}{2.0}} = 1.1 \cdot e^{0.9} = 2.65$$

in the direction of the motion, i.e., toward the front.
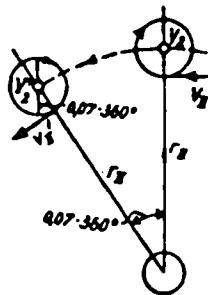
Figure 25a.

After the 464 days, necessary for the 5.93 orbits, an equal radiation of $(\frac{m_o}{m_1})_{II}$ = 2.65 must be used to withdraw the vehicle in opposite direction from the gravitational field of Venus and to return it into its own elliptic orbit, in which after a further 146 days it returns to the neighborhood of Earth. At the instant of the crossing, again assumed to occur r_3 = 800,000 km from the Earth center, the relative velocity with respect to Earth has to be reduced by further mass radiation to the value of v_3 = 0.09 km/sec, established in section II. Since at this instant the velocity of the vehicle is v_I = 27.3 km/sec and the speed of the Earth in its trajectory is v_e = 29.7 km/sec, the necessary increase in velocity is

$$\Delta v_I' = 29.7 - 27.3 - 0.09 = \sim 2.3 \text{ km/sec}$$

84

and must now be effected, so as to accelerate the vehicle, i.e., toward the rear:

$$\left(\frac{m_o}{m_1}\right)_I' = \nu \cdot e^{\frac{2.3}{2.0}} = 1.1 \cdot e^{1.15} = 3.47.$$

The entire journey in this case lasts -- including the 30 days required for ascent and launching:

$$30 + 146 + 464 + 146 = 786 \text{ days} = 2.15 \text{ years}.$$

If m_1 denotes the mass of the returning vehicle, m_o the total mass at the beginning of the ascent, including fuel, then -- not taking into account the change of mass due to supplies being used up -- approximately:

$$\frac{m_o}{m_1} = 933 \cdot 3.65 \cdot 2.65^2 \cdot 3.47 = 83,000.$$

2nd possibility (see Figure 26): From the point V_2 the vehicle is to return to Earth E_4 not directly, but by detour via F_3. The coincidence with Earth can happen at best 1.5 Earth years after separation at E_1. The Sun distance r_{III} of point F_3 is to be chosen therefore such, that the entire travel time from E_1 via V_2 and F_3 to E_4 takes 547.5 days. The total time T is composed out of the times T_1 and T_2 and T_3, required to cover the 3 half ellipses I, II and III with the major semi-axes

$$a_1 = \frac{r_I + r_{II}}{2} = 128,500,000 \text{ km};$$

$$a_2 = \frac{r_{II} + r_{III}}{2}; \quad a_3 = \frac{r_{III} + r_I}{2}.$$

From these two expressions:

$$a_3 - a_2 = \frac{r_I - r_{II}}{2} = \frac{149,000,000 - 108,000,000}{2} = 20,500,000 \text{ km}.$$

Further

$$T_3 + T_2 = T - T_1 = 547.5 - 146 = 401.5 \text{ days},$$

or by (48) -- for half an elliptic orbit --

$$\pi \sqrt{\frac{a_3^3}{\mu}} + \pi \sqrt{\frac{a_2^3}{\mu}} = 401.5 \text{ days} = 34,700,000 \text{ sec},$$

or

$$\sqrt{a_3^3} + \sqrt{a_2^3} = \frac{34,700,000}{\pi} \cdot \sqrt{\mu} = \frac{34,700,000}{\pi} \sqrt{132,000,000,000,000};$$

therefore

$$\sqrt{a_3^3} + \sqrt{a_2^3} = 4,010,000,000,000,000,$$

and

$$a_3 - a_2 = 20,500,000.$$

These two equations are satisfied by:

$$a_3 = 169,000,000 \text{ km and } a_2 = 148,500,000 \text{ km.}$$

Therefore from $a_2 = \dfrac{r_{II} + r_{III}}{2}$:

$$r_{III} = 2a_2 - r_{II} = 297,000,000 - 108,000,000 = 189,000,000 \text{ km.}$$

Departure at E_1 occurred with a velocity $v_I = 27.3$ km/sec and arrival at V_2 with a velocity:

$$v_{II} = v_I \cdot \frac{r_I}{r_{II}} = 27.3 \cdot \frac{149}{108} = 37.6 \text{ km/sec.}$$

The velocity required at V_2, to reach F_3, is by (49):

$$v_{II}' = \sqrt{\frac{2\mu}{r_{II} + r_{III}} \cdot \frac{r_{III}}{r_{II}}} = \sqrt{\frac{264,000}{297} \cdot \frac{189}{108}} = 39.4 \text{ km/sec;}$$

which determines the velocity of arrival at F_3:

$$v_{III} = v_{II}' \cdot \frac{r_{II}}{r_{III}} = 39.4 \cdot \frac{108}{189} = 22.5 \text{ km/sec.}$$

The departure velocity necessary at F_3, to attain E_4, is

$$v_{III}' = \sqrt{\frac{2\mu}{r_{III} + r_I} \cdot \frac{r_I}{r_{III}}} = \sqrt{\frac{264,000}{338} \cdot \frac{149}{189}} = 24.8 \text{ km/sec,}$$

and finally the velocity, arriving at E_4

$$v_{IV} = v_{III}' \cdot \frac{r_{III}}{r_I} = 24.8 \cdot \frac{189}{149} = 31.5 \text{ km/sec}$$

compared to the Earth velocity

$$v_e = 29.7 \text{ km/sec.}$$

Accordingly the following velocity changes are necessary:

at departure E_1: $\Delta v_I = 27.3 - 29.7 = -2.4$ km/sec,

at passing V_2: $\Delta v_{II} = 39.4 - 37.6 = +1.8$ km/sec,

at passing F_3: $\Delta v_{III} = 24.8 - 22.5 = +2.3$ km/sec,

at arrival E_4: $\Delta v_{IV} = 29.7 - 31.5 + 0.09 = -1.7$ km/sec

(initiating landing).

The masses necessary to attain these velocity changes, using $c = 2.0$ km/sec, are given in sequence by

$$\left(\frac{m_0}{m_1}\right)_I = \nu \cdot e^{\frac{2.4}{2.0}} = 1.1 \cdot e^{1.20} = 3.65$$

$$\left(\frac{m_0}{m_1}\right)_{II} = \nu \cdot e^{\frac{1.8}{2.0}} = 1.1 \cdot e^{0.90} = 2.71$$

$$\left(\frac{m_0}{m_1}\right)_{III} = \nu \cdot e^{\frac{2.3}{2.0}} = 1.1 \cdot e^{1.15} = 3.47$$

$$\left(\frac{m_0}{m_1}\right)_{IV} = \nu \cdot e^{\frac{1.7}{2.0}} = 1.1 \cdot e^{0.85} = 2.57$$

these are to be directed forward at E_1 and E_4 and backwards at V_2 and F_3

With the same notation as before

$$\frac{m_0}{m_1} = 933 \cdot 3.65 \cdot 2.71 \cdot 3.47 \cdot 2.57 = 82,000.$$

The entire journey takes

$$30.5 + 547.5 = 578 \text{ days} = 1.58 \text{ years,}$$

including landing and ascent.

Of these possibilities, the second one for the same fuel consumption has the advantage of a shorter travel time, while the first permits a longer stay in the neighborhood of Venus.

A visit to Mars would take a similar form. Here however its position at the instant of proximity would have to be more accurately calculated, since its orbit is more eccentric than that of Earth or Venus, its Sun distance varying between 248,000,000 km

and 205,000,000 km. Now the detour via F_3, according to Figure 26, at its apogee r_{III} equals 189,000,000 km, is nearly equal to the smallest distance of Mars to the Sun, leaving only 16,000,000 km. With a suitable choice of the time of ascent during a mutual constellation of Earth, Venus and Mars and with a suitable adjustment of r_{II} and r_{III}, a passage at relatively small distance (about $\frac{16}{2}$ = 8 million km each time) of Mars as well as Venus can be achieved in a single journey of about 1 1/2 years' duration.
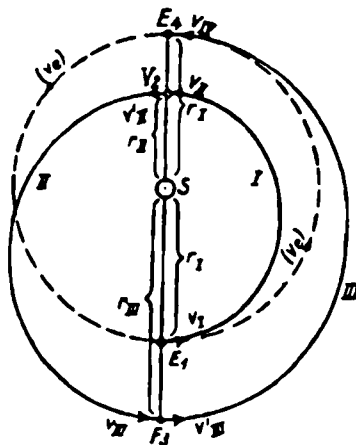
Figure 26.

This 580 day journey would not quite take 20 times as much as the 30 day journey into space, described in section III. For a rough estimate of the vehicle mass now required, we multiply by 20 the parts of the weight, which depend on the duration and are denoted on pages 67-68 under b), c), d), e), while those independent of time, i.e., listed under a), f), g) and i), are left as is. With regard to the larger weight h), required by the larger storage space, we allow 3 times its previous weight. Since with the storage space also the heat-transferring surface increases, we imply here a better insulation. These assumptions lead to a weight less fuel

$(240 + 60 + 200 + 140) \cdot 20$ = 12,800 kg

$+ 200 + 200 + 240 + 200 + 740$ =   1,580 kg

$+ 780 \cdot 3$ =   2,340 kg

total of 16,720 kg = 16.72 t.

Between E_1 and V_2, T_1 = 146 days pass, between V_1 and F_3

$$T_2 = T_1 \cdot \sqrt{\frac{a_2{}^3}{a_1{}^3}} = 146 \sqrt{\frac{148.5^3}{128.5^3}} = 181 \text{ days};$$

between F_3 and E_4

$$T_3 = T_1 \cdot \sqrt{\frac{a_3{}^3}{a_1{}^3}} = 146 \sqrt{\frac{169.0^3}{128.5^3}} = 220 \text{ days}.$$

Of the 12.8 tons of supplies, consumed were:

during the 15 days ascent up to E_1: $\quad 12.8 \cdot \dfrac{15}{578} = 0.33$ t,

between E_1 and V_2: $\quad 12.8 \cdot \dfrac{146}{578} = 3.20$ t,

between V_2 and F_3: $\quad 12.8 \cdot \dfrac{181}{578} = 3.95$ t,

between F_3 and E_4: $\quad 12.8 \cdot \dfrac{220}{578} = 4.80$ t,

between departure and E_4 therefore 12.28 tons.

After arrival at E_4, the weight of the vehicle remains 16.72 − 12.28 = 4.44 tons.

Immediately before arrival at E_4, the total mass is	4.44 · 2.57 =	11.40 t
after arrival at F_3	11.40 + 4.80 =	16.20 t
immediately before arrival at F_3	16.20 · 3.47 =	56.30 t
after arrival at V_2	56.30 + 3.95 =	60.25 t
immediately before arrival at V_2	60.25 · 2.71 =	163.00 t
after arrival at E_1	163.00 + 3.20 =	166.20 t
immediately before arrival at E_1	166.20 · 3.65 =	606.67 t
after acceleration has ended	606.67 + 0.33 =	607 t
at departure G_0 =	607 · 933 = 567,000	t

or abbreviated:

$$G_0 = \left[\left\{ \left[(4.44 \cdot 2.57 + 4.8) \cdot 3.47 + 3.95 \right] \cdot 2.71 + 3.2 \right\} \cdot 3.65 + 0.33 \right] \cdot 933 = 567,000 \text{ t}.$$

The main part of the ammunition, to be taken along, is taken up by the fuel required for the initial acceleration, but such fuel is also necessary (say 607 - 17 = 590 tons), to change the velocity during the journey and such a mass will present difficulties as regards maneuverability. How much G_0 depends upon the velocity, with which the mass is radiated, c, is clarified by the following list of values for G_0, resulting for different values of c for a constant acceleration of αc = 30 m/sec^2:

c = 2 km/sec: $G_0 = [\{[(4.44 \cdot 2.57 + 4.8) \cdot 3.47 + 3.95] \cdot 2.71 +$

$3.2\} \cdot 3.65 + 0.33] \cdot 933 = 567,000$ t

c = 2.5 km/sec: $G_0 = [\{[(4.44 \cdot 2.17 + 4.8) \cdot 2.77 + 3.95] \cdot$

$2.27 + 3.2\} \cdot 2.87 + 0.33] \cdot 235 = 69,500$ t

c = 3 km/sec: $G_0 = [\{[(4.44 \cdot 1.95 + 4.8) \cdot 2.38 + 3.95] \cdot 2.00 +$

$3.2\} \cdot 2.45 + 0.33] \cdot 95 = 17,600$ t

c = 4 km/sec: $G_0 = [\{[(4.44 \cdot 1.69 + 4.8) \cdot 1.98 + 3.95] \cdot 1.73 +$

$3.2\} \cdot 2.00 + 0.33] \cdot 30 = 3,150$ t

c = 5 km/sec: $G_0 = [\{[(4.44 \cdot 1.55 + 4.8) \cdot 1.75 + 3.95] \cdot 1.57 +$

$3.2\} \cdot 1.78 + 0.33] \cdot 15 = 1,130$ t.

V

LANDING ON OTHER CELESTIAL OBJECTS

Of the planets, Venus seems to be particularly suited for a landing, because, presumably, it has an atmosphere similar to that of Earth. This and the further assumption of similar gravitation conditions would accordingly permit a landing exactly as described in sections II and III for Earth. It could begin by imparting to the vehicle at the distance r_3 = 800,000 km from the center of Venus a tangential velocity v_3 = 0.09 km/sec (see Figure 14). (Compare what has been said about Mars and Venus on pages 88-89. Since furthermore the atmosphere of Venus is very high and dense, landing should be simpler than on Earth.) The preceding journey proceeds exactly as determined for the segment E_1 - V_2, following Figure 25, i.e., V_2 is passed with a velocity v_{II} = 37.6 km/sec, compared with a velocity of Venus v_v = 35.1 km/sec; the relative velocity at that instant being 37.6 - 35.1 = 2.5 km/sec. To reduce it to 0.09 km/sec, a reduction by approximately Δv_{II} = 2.4 km/sec is necessary, requiring a mass

$$\left(\frac{m_0}{m_1}\right)_{II} = \nu \cdot e^{\frac{\Delta v_{II}}{c}} = 1.1 \cdot e^{\frac{2.4}{2.0}} = 1.1 \cdot e^{1.2} = 3.65,$$

while at E_0 as before

$$\left(\frac{m_0}{m_1}\right)_I = 3.65.$$

The travel time is now:

ascent at E_1	15 days
Sun-centered orbit $E_1 - V_2$	146 days
landing at V_2	<u>15 days</u>
total	176 days,

i.e., about 6 times the 30 days, discussed in section III. In determining the mass, the parts of the weight, b), c), d), e) may be multiplied by 6, while a), f), g), i) are taken as is and the vehicle weight h) is about doubled, giving a total weight (less fuel) of

$$(240 + 60 + 200 + 140) \cdot 6 \quad = 3,860$$

$$+ \ 200 + 200 + 240 + 200 + 740 = 1,580$$

$$+ \ 780 \cdot 2 \qquad\qquad\qquad = \underline{1,560}$$

$$\text{total} = 7,000 \text{ kg} = 7.0 \text{ t.}$$

Of the supplies, used up, were, as before:

between departure and E_1	0.3 t
between E_1 and V_2	<u>3.2 t</u>
therefore between departure and V_2	3.5 t,

therefore after arrival at V_2 a weight of: $7.0 - 3.5 = 3.5$ t remains. The total weight at the ascent from Earth is therefore as follows:

for $c = 2$ km/sec: $G_0 = \left[(3.5 \cdot 3.65 + 3.2) \cdot 3.65 + 0.3\right] \cdot 933 =$

54,800 t

for $c = 2.5$ km/sec: $G_0 = \left[(3.5 \cdot 2.87 + 3.2) \cdot 2.87 + 0.3\right] \cdot 235 =$

8,800 t

for c = 3 km/sec: $G_o = \underline{/}(3.5 \cdot 2.45 + 3.2) \cdot 2.45 + 0.3\underline{/} \cdot 95 =$

2,800 t

for c = 4 km/sec: $G_o = \underline{/}(3.5 \cdot 2.00 + 3.2) \cdot 2.00 + 0.3\underline{/} \cdot 30 =$

620 t

for c = 5 km/sec: $G_o = \underline{/}(3.5 \cdot 1.78 + 3.2) \cdot 1.78 + 0.3\underline{/} \cdot 15 =$

260 t.

For a self-motivated return to Earth from Venus, the same ascent weight is necessary, if however the mass necessary for the return trip has to be taken along immediately, then the following weight at departure would be necessary:

for c = 2 km/sec: $54,800 \cdot 3.65^2 \cdot 933 = 670,000,000$ t

for c = 2.5 km/sec: $8,800 \cdot 2.87^2 \cdot 235 = 17,000,000$ t

for c = 3 km/sec: $2,800 \cdot 2.45^2 \cdot 95 = 1,600,000$ t

for c = 4 km/sec: $620 \cdot 2.00^2 \cdot 30 = 74,000$ t

for c = 5 km/sec: $260 \cdot 1.78^2 \cdot 15 = 1,240$ t.

A landing on Venus therefore presupposes that the fuel necessary for a return, may be manufactured by simple means of raw materials available there.

A landing on Mars cannot be undertaken after the fashion described for Venus or Earth because presumably the atmosphere there is lacking. Retardation must now be effected in a manner inverse to acceleration, described in section I. Mars has a radius of $r_o = 3,373$ km, the gravitational acceleration -- as determined from the motion of its 2 moons -- is $g_o = 3.7$ m/sec^2 = 0.0037 km/sec^2.

If again an acceleration of $c\alpha = 0.03$ km/sec^2 and an exhaust velocity c = 2 km/sec is assumed, so that $\alpha = \frac{c\alpha}{c} = \frac{0.03}{2.0} = \frac{0.015}{sec}$, then the distance r_1, at which the acceleration must commence, is by (7):

$$r_1 = r_o(1 + \frac{g_o}{c\alpha}) = 3,392 (1 + \frac{0.0037}{0.03}) = 3,800 \text{ km}$$

and the velocity at arrival in r_1 from a large distance is by (8):

$$v_1 = \sqrt{\frac{2g_0r_0^2}{r_1}} = \sqrt{\frac{2 \cdot 0.0037 \cdot 3.392^2}{3,800}} = 4.70 \text{ km/sec};$$

further the average retardation by (9):

$$\beta = c\alpha - \frac{g_0}{3}\left(2 + \frac{r_0^2}{r_1^2}\right) = 0.03 - \frac{0.0037}{3}\left(2 + \frac{3.392^2}{3,800^2}\right) = 0.02655 \text{ km/sec}^2,$$

the approximate time therefore for retardation by (10):

$$t_1 = \frac{v_1}{\beta} = \frac{4.70}{0.02655} = 177 \text{ sec},$$

and the mass ratio of the radiation by (11):

$$\frac{m_0}{m_1} = e^{\alpha t_1} = e^{0.015 \cdot 177} = e^{2.66} = 14.3.$$

If $r_I = 149,000,000$ km denotes the Earth-Sun distance, and if Mars is to be reached at its perigee $r_{II} = 205,000,000$ km, then the vehicle, after leaving Earth, must receive a tangential velocity, according to (49), of

$$v_I = \sqrt{\frac{264,000}{354} \cdot \frac{205}{149}} = 32.0 \text{ km/sec}$$

compared to the Earth velocity of 29.7 km/sec, while Mars is passed with a velocity of

$$v_{II} = 32.0 \cdot \frac{149}{205} = 23.2 \text{ km/sec}$$

compared to a velocity in its orbit around the Sun of 26.5 km/sec. The necessary changes in velocity are therefore after leaving Earth

$$\Delta v_I = 32.0 - 29.7 = 2.3 \text{ km/sec}$$

with

$$\left(\frac{m_0}{m_1}\right)_I = \gamma \cdot e^{\frac{2.3}{2.0}} = 1.1 \cdot e^{1.15} = 3.47;$$

for landing on Mars:

$$\Delta v_{II} = 26.5 - 23.2 = 3.3 \text{ km/sec}$$

with

$$\left(\frac{m_o}{m_1}\right)_{II} = \gamma \cdot e^{\frac{3.3}{2.0}} = 1.1 \cdot e^{1.65} = 5.73.$$

The travel time is composed as follows:

ascent from Earth about   15 days

Sun-centered journey Earth-Mars: $\pi \cdot \sqrt{\dfrac{a^3}{\mu}}$

$$a = \frac{r_I + r_{II}}{2} = 177,000,000 \text{ km and}$$

$$\mu = 132,000,000,000 \frac{km^3}{sec^2} \quad \text{therefore}$$

$$\pi \sqrt{\frac{177,000,000^3}{132,000,000,000}} = 20,350,000 \text{ sec} = \qquad 235 \text{ days}$$

landing on Mars approximately   <u>15 days</u>

<div align="right">total: 265 days</div>

i.e., about 9 times the 30 day journey of section III. Similar to the journey around Venus, the initial weight of the vehicle, less fuel, may be found by:

$$\frac{9}{6} \cdot 3,860 + 1,580 + 1,560 = 5,790 + 3,140 = 8,930 \text{ kg} = \sim 9 \text{ t}.$$

Of the approximately 5.8 t supplies consumed were:

during the ascent from Earth $\dfrac{15}{265} \cdot 5.8 = \sim 0.3$ t

during the Sun-centered journey Earth-Mars $\dfrac{235}{265} \cdot 5.8 = \quad 5.2$ t

during the landing on Mars ~ 0.3 t

At arrival on Mars there are left 9.0 - 5.8 = 3.2 t, and the total weight at the beginning of the ascent is

for c = 2 km/sec: $G_o = \left\{ [(3.2 \cdot 14.3 + 0.3) \cdot 5.73 + 5.2] \cdot 3.47 + \right.$

$\left. 0.3 \right\} \cdot 933 = 875,000$ t

for $c = 2.5$ km/sec: $G_o = \{\underline{/}(3.2 \cdot 8.3 + 0.3) \cdot 4.13 + 5.\underline{2}\underline{/} \cdot 2.77 + 0.3\} \cdot 235 = 76{,}500$ t

for $c = 3$ km/sec: $G_o = \{\underline{/}(3.2 \cdot 5.9 + 0.3) \cdot 3.32 + 5.\underline{2}\underline{/} \cdot 2.38 + 0.3\} \cdot 95 = 15{,}600$ t

for $c = 4$ km/sec: $G_o = \{\underline{/}(3.2 \cdot 3.8 + 0.3) \cdot 2.51 + 5.\underline{2}\underline{/} \cdot 1.98 + 0.3\} \cdot 30 = 2{,}200$ t

for $c = 5$ km/sec: $G_o = \{\underline{/}(3.2 \cdot 2.9 + 0.3) \cdot 2.14 + 5.\underline{2}\underline{/} \cdot 1.75 + 0.3\} \cdot 15 = 690$ t

thus very much less favorable than in the case of Venus, which has
an atmosphere. Much more favorable however is the self-propelled
return to Earth from Mars -- again of course under the assumption,
that fuel can be found and processed there -- in this case the fac-
tor 933, etc., is eliminated because of the Earth atmosphere and
the remaining factors in the inverse sequence, corresponding to a
return trip, are

for $c = 2$ km/sec: $G_o = \{\underline{/}(3.2 + 0.3) \cdot 3.47 + 5.\underline{2}\underline{/} \cdot 5.73 + 0.3\} \cdot 14.3 = 1{,}430$ t

for $c = 2.5$ km/sec: $G_o = \{\underline{/}(3.2 + 0.3) \cdot 2.77 + 5.\underline{2}\underline{/} \cdot 4.13 + 0.3\} \cdot 8.3 = 515$ t

for $c = 3$ km/sec: $G_o = \{\underline{/}(3.2 + 0.3) \cdot 2.38 + 5.\underline{2}\underline{/} \cdot 3.32 + 0.3\} \cdot 5.9 = 265$ t

for $c = 4$ km/sec: $G_o = \{\underline{/}(3.2 + 0.3) \cdot 1.98 + 5.\underline{2}\underline{/} \cdot 2.51 + 0.3\} \cdot 3.8 = 118$ t

for $c = 5$ km/sec: $G_o = \{\underline{/}(3.2 + 0.3) \cdot 1.75 + 5.\underline{2}\underline{/} \cdot 2.14 + 0.3\} \cdot 2.9 = 71$ t.

A landing on the Moon would be similar to one on Mars. Here,
using the same notation, we have:

r_o = 1,740 km; g_o = 0.0016 km/sec^2 (since the density of the Moon is lower than Earth's, we have g_o < 0.0098 · $\frac{1,740}{6,380}$); αc = 0.03 km/sec^2; c = 2.0 km/sec; α = $\frac{0.015}{sec}$; r_1 = 1,740 (1 + $\frac{0.0016}{0.03}$) = 1,830 km; $v_1 \sqrt{\frac{2 \cdot 0.0016 \cdot 1,740^2}{1,830}}$ = 2.30 km/sec; β = ~0.03 − $\frac{0.0016}{3}$ (2 + $\frac{1,740^2}{1,830^2}$) = 0.0284 km/sec^2; t_1 = $\frac{v_1}{\beta}$ = $\frac{2.30}{0.0284}$ = 81 sec;

$\frac{m_o}{m_1}$ = $e^{\alpha t_1}$ = $e^{0.015 \cdot 81}$ = $e^{1.22}$ = 3.40.

Since the duration of the journey in this case is at most twice as long as that required for the extended space journey of section III, requiring a correspondingly smaller stock of supplies, one may now assume a weight, less fuel, averaging 2.6 t in place of 3 t, giving an initial weight for the one-way trip Earth-Moon:

for c = 2 km/sec: G_o = 2.6 · 3.4 · 933 = 8,250 t

for c = 2.5 km/sec: G_o = 2.6 · 2.64 · 235 = 1,610 t

for c = 3 km/sec: G_o = 2.6 · 2.25 · 95 = 555 t

for c = 4 km/sec: G_o = 2.6 · 1.85 · 30 = 144 t

for c = 5 km/sec: G_o = 2.6 · 1.64 · 15 = 64 t

and the initial weight for the return trip Moon-Earth:

for c = 2 km/sec: G_o = 2.6 · 3.4 = 8.9 t

for c = 2.5 km/sec: G_o = 2.6 · 2.64 = 6.9 t

for c = 3 km/sec: G_o = 2.6 · 2.25 = 5.9 t

for c = 4 km/sec: G_o = 2.6 · 1.85 = 4.8 t

for c = 5 km/sec: G_o = 2.6 · 1.64 = 4.3 t.

If on the other hand the return to Earth is to be assured, then the weight at departure from Earth must be:

for c = 2 km/sec: G_o = 2.6 · 3.4^2 · 933 = 28,000 t

for c = 2.5 km/sec: G_o = 2.6 · 2.64^2 · 235 = 4,250 t

for c = 3 km/sec: $G_0 = 2.6 \cdot 2.25^2 \cdot 95 = 1,250$ t

for c = 4 km/sec: $G_0 = 2.6 \cdot 1.85^2 \cdot 30 = 890$ t

for c = 5 km/sec: $G_0 = 2.6 \cdot 1.64^2 \cdot 15 = 700$ t.

The relative case of the attainability of the Moon and the low mass ratio $\frac{m_0}{m_1} = 4.0$ at departure from the Moon leads to the concept of using the Moon as an intermediate station for further undertakings. A condition here is that the necessary fuel may be gained from the Moon itself, i.e., an explosives factory can be constructed there. To investigate this possibility, an investigative 2-way trip with assured return, f.i. one, involving $G_0 = 38,000$ t at c = 2 km/sec, would have to be carried out, which after all is not completely outside the range of possibility. In case of positive findings, every further Moon journey would require only 8,250 t and each return from Moon to Earth only 8.9 t, and for each trip to the planets, initiated from Moon rather than from Earth, the mass ratio required would be $\frac{m_0}{m_1} = 3.4$ t , etc., in place of the Earth-mass ratio $\frac{m_0}{m_1} = 933$ t , etc., whereby however the return trip would be directly to Earth, because of the more favorable landing conditions.

The following initial weights would be necessary:

a) round-trip Moon-Venus-Mars-Earth (no intermediate landing on Venus or Mars):

for c = 2 km/sec: $G_0 = \frac{3.4}{933} \cdot 567,000 = 2,070$ t

for c = 2.5 km/sec: $G_0 = \frac{2.64}{235} \cdot 69,500 = 780$ t

for c = 3 km/sec: $G_0 = \frac{2.25}{95} \cdot 17,600 = 417$ t

for c = 4 km/sec: $G_0 = \frac{1.85}{30} \cdot 3,150 = 194$ t

for c = 5 km/sec: $G_0 = \frac{1.64}{15} \cdot 1,130 = 124$ t

b) for a trip Moon-Mars, including landing, but without insuring a return:

for c = 2 km/sec: $G_0 = \dfrac{3.4}{933} \cdot 875{,}000 = 3{,}190$ t

for c = 2.5 km/sec: $G_0 = \dfrac{2.64}{235} \cdot 76{,}500 = 860$ t

for c = 3 km/sec: $G_0 = \dfrac{2.25}{95} \cdot 15{,}600 = 370$ t

for c = 4 km/sec: $G_0 = \dfrac{1.85}{30} \cdot 2{,}200 = 136$ t

for c = 5 km/sec: $G_0 = \dfrac{1.64}{15} \cdot 690 = 76$ t

c) for a trip Moon-Venus, including landing, but without insuring a return:

for c = 2 km/sec: $G_0 = \dfrac{3.4}{933} \cdot 54{,}800 = 200$ t

for c = 2.5 km/sec: $G_0 = \dfrac{2.64}{235} \cdot 8{,}800 = 99$ t

for c = 3 km/sec: $G_0 = \dfrac{2.25}{95} \cdot 2{,}800 = 67$ t

for c = 4 km/sec: $G_0 = \dfrac{1.85}{30} \cdot 620 = 38$ t

for c = 5 km/sec: $G_0 = \dfrac{1.64}{15} \cdot 260 = 29$ t

d) for a landing on Mars, safeguarding the return trip (safe for an initial investigation), using $\dfrac{m_0}{m_1} = 14.3$, etc., and taking into consideration a further 5.8 t of supplies for the return trip:

for c = 2 km/sec: $G_0 = 3{,}190 \cdot 14.3 \cdot \dfrac{9 + 5.8}{9} = 75{,}000$ t

for c = 2.5 km/sec: $G_0 = 860 \cdot 8.3 \cdot \dfrac{9 + 5.8}{9} = 11{,}800$ t

for c = 3 km/sec: $G_0 = 370 \cdot 5.9 \cdot \dfrac{9 + 5.8}{9} = 3{,}600$ t

for c = 4 km/sec: $G_0 = 136 \cdot 3.8 \cdot \dfrac{9 + 5.8}{9} = 850$ t

for c = 5 km/sec: $G_o = 76 \cdot 2.9 \cdot \dfrac{9 + 5.8}{9} = 360$ t

e) for a landing on Venus, safeguarding a return in a suitable way:

for c = 2 km/sec: $G_o = 200 \cdot 933 \cdot \dfrac{7 + 3.9}{7} = 290{,}000$ t

for c = 2.5 km/sec: $G_o = 99 \cdot 235 \cdot \dfrac{7 + 3.9}{7} = 36{,}300$ t

for c = 3 km/sec: $G_o = 67 \cdot 95 \cdot \dfrac{7 + 3.9}{7} = 9{,}900$ t

for c = 4 km/sec: $G_o = 38 \cdot 30 \cdot \dfrac{7 + 3.9}{7} = 1{,}780$ t

for c = 5 km/sec: $G_o = 29 \cdot 15 \cdot \dfrac{7 + 3.9}{7} = 680$ t.

Safeguarding the return in case e) is more difficult than in case d), nevertheless and in spite of the fact that a self-propelled return from Venus (with nearly the same values G_o as at departure from Earth) could only be accomplished with a high exhaust velocity c; the probability of finding there an atmosphere and consequently conditions for life, similar to those on Earth, is so great, and the difficulties of the one-way trip there -- having once established the station on Moon -- so small that presumably Venus must be primarily considered as a goal for immigration, Mars on the other hand a goal for scientific investigation trips.

During departure from Moon, one should, strictly speaking, take into account the velocity of the Moon around the Earth, as was done (Figure 16) with the rotation of the Earth; its influence will however not here be investigated.

For simplicity, we have up to now only discussed those connecting elliptic segments between planets, which touch the 2 planets, which are to be connected, i.e., those requiring changes in velocity, but no changes in direction. It is not obvious that these tangential ellipses constitute the most favorable connection. Rather it is conceivable that other ellipses, intersecting planetary orbits, would be more expeditious, since without doubt they would provide shorter connections. For this reason we examine first the other limit of a change in direction at constant velocity.

The connecting ellipse to be found must cross both planetary orbits with velocities equal to the velocity of those planets; using the notation of Figure 27 we have for the connecting ellipses, according to (41):

$$1. \quad v_a^2 - \frac{2\mu}{r_a} = v_1^2 - \frac{2\mu}{r_1} ;$$

$$2. \quad v_a^2 - \frac{2\mu}{r_a} = v_2^2 - \frac{2\mu}{r_2} ;$$

and for the circular orbits r_1 and r_2, by (37):

$$v_1^2 = \frac{\mu}{r_1} ;$$

$$v_2^2 = \frac{\mu}{r_2} ;$$

therefore we would require:

$$1. \quad v_a^2 - \frac{2\mu}{r_a} = \frac{\mu}{r_1} - \frac{2\mu}{r_1} ;$$

$$2. \quad v_a^2 - \frac{2\mu}{r_a} = \frac{\mu}{r_2} - \frac{2\mu}{r_2} ;$$

or

$$1. \quad \frac{2\mu}{r_a} - v_a^2 = \frac{\mu}{r_1} ;$$

$$2. \quad \frac{2\mu}{r_a} - v_a^2 = \frac{\mu}{r_2} .$$

The two equations are contradictory. It follows that the condition of crossing both orbits with corresponding velocities cannot be met.

If now only the condition is required, that one of the planetary orbits, say radius r_2, has velocity equal to that of the ellipse, then only one of the equations remains:

$$\frac{2\mu}{r_a} - v_a^2 = \frac{\mu}{r_2} ;$$

and choosing r_a arbitrarily:

$$v_a{}^2 = \frac{2\mu}{r_a} - \frac{\mu}{r_2} \; ;$$

further by (45):

$$a = \frac{\mu}{\frac{2\mu}{r_a} - v_a{}^2} = \frac{\mu}{\frac{\mu}{r_2}} = r_2 \; ,$$

and from (46):

$$b = \frac{v_a r_a}{\sqrt{\frac{2\mu}{r_a} - v_a{}^2}} = \frac{v_a r_a}{\sqrt{\frac{\mu}{r_2}}} = r_a \sqrt{\frac{2r_2}{r_a} - 1} \; ;$$

i.e., each ellipse, whose semi-major axis a equals the radius r_2 on a circular orbit, will cross this orbit with the corresponding velocity.
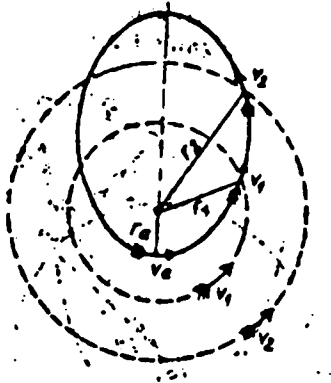

Figure 27.

The angle, subtended at the crossing by the two trajectories, which at the same time gives the tangential inclination of the trajectory, is by Figure 28:

$$\tan \alpha = \frac{dr}{r_2 d\varphi} = \frac{1}{r_2} \cdot \frac{dr}{d\varphi} \; ;$$

i.e., according to (43) with $r = r_2$:

$$\tan \alpha = \sqrt{\frac{v_a^2 - \frac{2\mu}{r_a}}{v_a^2 r_a^2} \cdot r_2^2 + \frac{2\mu}{v_a^2 r_a^2} \cdot r_2 - 1} \; ;$$

or, since in this case

$$v_a^2 - \frac{2\mu}{r_a} = -\frac{\mu}{r_2}$$

we have:

$$\tan \alpha = \sqrt{-\frac{\mu r_2}{v_a^2 r_a^2} + \frac{2\mu r_2}{v_a^2 r_a^2} - 1} = \sqrt{\frac{\mu r_2}{v_a^2 r_a^2} - 1} \; .$$
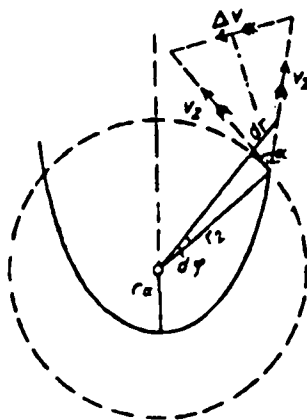

Figure 28.

Of the many possible connecting ellipses with semi-major axis $a = r_2$, we now wish to examine the one which touches the planetary orbit with radius r_1, i.e., the one which requires a change of direction only near one planet and a change in velocity only near the other one. For this we must choose

$$r_a = r_1$$

so that

$$v_a^2 = \frac{2\mu}{r_1} - \frac{\mu}{r_2} = \mu \cdot \frac{2r_2 - r_1}{r_1 r_2} \; ,$$

and

$$\tan \alpha = \sqrt{\frac{\mu r_2}{r_1^2 \cdot \mu \cdot \frac{2r_2 + r_1}{r_1 r_2}} - 1} = \sqrt{\frac{r_2^2}{r_1(2r_2 - r_1)} - 1} \; ;$$

or

$$\tan \alpha = \sqrt{\frac{r_2^2 - 2r_1 r_2 + r_1^2}{r_1(2r_2 - r_1)}} = \sqrt{\frac{(r_2 - r_1)^2}{r_1(2r_2 - r_1)}} \; .$$

To produce the change of direction without change in velocity v_2 at the intersection, a velocity component has to be added at right angles to the bisector of the angle α, having a magnitude

$$\Delta v = 2 \cdot v_2 \cdot \sin \frac{\alpha}{2} \text{ (see Figure 28).}$$

F.i., for an ellipse, touching the Earth orbit and intersecting that of Venus in the desired way, i.i. for

$$r_1 = 149,000,000 \text{ km}$$

$$r_2 = 108,000,000 \text{ km}$$

$$v_2 = 35.1 \text{ km/sec:}$$

$$\tan \alpha = \sqrt{\frac{(108 - 149)^2}{149 \cdot (216 - 149)}} = \frac{41}{\sqrt{149 \cdot 67}} = 0.41;$$

$\alpha = \sim 22 \; 1/4^\circ$; $\Delta v = 2 \cdot 35.1 \cdot \sin 11 \; 1/8^\circ = 13.5 \text{ km/sec};$

for the ellipse, touching the orbit of Venus and cutting the orbit of Earth, i.e., for

$$r_1 = 108,000,000 \text{ km}$$

$$r_2 = 149,000,000 \text{ km}$$

$$v_2 = 29.7 \text{ km/sec:}$$

$$\tan \alpha = \sqrt{\frac{(149 - 108)^2}{108 \; (298 - 108)}} = \frac{41}{\sqrt{108 \cdot 190}} = 0.286;$$

$\alpha = \sim 16^\circ$; $\Delta v = 2 \cdot 29.7 \cdot \sin 8^\circ = 8.3 \text{ km/sec};$

for the ellipse, touching the Earth orbit and intersecting that
of Mars, i.e., for

$$r_1 = 149,000,000 \text{ km}$$

$$r_2 = 205,000,000 \text{ km (circular orbit assumed)}$$

$$v_2 = 26.5 \text{ km/sec:}$$

$$\tan \alpha = \sqrt{\frac{(205 - 149)^2}{149 \, (410 - 149)}} = \frac{56}{\sqrt{149 \cdot 261}} = 0.284;$$

$\alpha = \sim 16°$; $\Delta v = 2 \cdot 26.5 \cdot \sin 8° = 7.4 \text{ km/sec;}$

for the ellipse, touching the Mars orbit and intersecting that of
Earth, i.e., for

$$r_1 = 205,000,000 \text{ km}$$

$$r_2 = 149,000,000 \text{ km}$$

$$v_2 = 29.7 \text{ km/sec:}$$

$$\tan \alpha = \sqrt{\frac{(149 - 205)^2}{205 \, (298 - 205)}} = \frac{56}{\sqrt{205 \cdot 93}} = 0.405;$$

$\alpha = \sim 22°$; $\Delta v = 2 \cdot 29.7 \cdot \sin 11° = 11.4 \text{ km/sec.}$

One can see that in all cases the velocity change Δv is much
larger than that required for an orbit touching both planetary
orbits. F.i., in the most favorable case (touching the Earth orbit
and crossing that of Mars), we have a $\Delta v = 7.4$ km/sec (in place of
$\Delta v_{II} = 3.3$ km/sec as on page 98), requiring a mass expenditure

$\frac{m_0}{m_1} = \nu \cdot e^{\frac{\Delta v}{c}}$ of the following values:

for c = 2 km/sec: $\frac{m_0}{m} = 1.1 \cdot e^{\frac{7.4}{2.0}} = 44.5$ in place of 5.73

for c = 2.5 " $\frac{m_0}{m} = 1.1 \cdot e^{\frac{7.4}{2.5}} = 21.4$ in place of 4.13

for c = 3 km/sec: $\frac{m_0}{m} = 1.1 \cdot e^{\frac{7.4}{3.0}} = 14.1$ in place of 3.32

for c = 4 km/sec: $\dfrac{m_0}{m} = 1.1 \cdot e^{\frac{7.4}{4.0}} = 7.05$ in place of 2.51

for c = 5 km/sec: $\dfrac{m_0}{m} = 1.1 \cdot e^{\frac{7.4}{5.0}} = 4.85$ in place of 2.14.

It is to be added that also during the transit from the planetary orbit, touched tangentially into the connecting ellipse, cutting the other planetary orbit, one requires a larger velocity change Δv_{II} than in the case of an orbit, touching both planetary orbits, since in the latter case the changing curvature has a minimum.

From these results it may be concluded that the ellipse, touching both planetary orbits, does in fact represent the most favorable trajectory.

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This work (originally published in 1925) contributes
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Treated are problems associated with leaving the
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