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MECHANICS

ON FUEL CONSUMPTION BY A ROCKET CROSSING THE ATMOSPHERE WITH CONSTANT ACCELERATION

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The computation method previously developed by the author for certain particular cases of rockets crossing the atmosphere [(1), p. 193—206] is generalized in the present paper, whose purpose it is to show that the amount of fuel required to overcome the resistance of the air to the flight of a rocket moving with constant acceleration away from the Earth surface, is defined by the following formula:

$$m = \frac{C\ S\Gamma}{\delta \sin\beta} K\,(182.844 + 79.580\,K + 21.665K^2 + 4.618K^3 + 0.843K^4 + 0.138K^5) \quad (1)$$

Here β is the angle of start of the rocket with reference to the horizon; Γ is the acceleration of its motion in m/sec.^{2}; γ is the perceivable acceleration due to the reactive force, *i. e.* that which would be recorded by an acceleration indicator placed within the rocket, in m/sec.^{2}; c is the velocity of the outflowing gases relative to the rocket (jet velocity), in km/sec.^{2}; C_x is the aerodynamic coefficient of resistance; S is the maximum cross-section of the rocket perpendicular to the direction of flight in m^2; m is the mass of the fuel in kg, and

$$K = \frac{0.1\gamma}{c\sqrt{0.1\Gamma \sin\beta}} \quad (2)$$

As, indeed, we have $v = \sqrt{2\Gamma s}$ for the velocity of a rocket moving with constant acceleration (s is the distance travelled by the rocket), the Galileo-Newton formula for air resistance takes the form

$$R = C_x \delta s S \Gamma \quad (3)$$

where δ is the density of the air.

The mass of the fuel consumed in overcoming the resistance of the air during an interval Δt when the jet velocity is c, will be

$$\Delta m_0 = \frac{R}{c}\,\Delta t \quad (4)$$

The acceleration of the rocket would be reduced by the added weight of the «main» store of fuel m_0 necessary to overcome the resistance of the air. To avoid this reduction in acceleration an additional amount of fuel m_1 is required in order to overcome the inertia of the «main» mass m_0. Again, for overcoming the inertia of the mass m_1 one has to provide the rocket with another mass of fuel m_2, and so on.

The added consumption of fuel during the interval Δt is

$$\Delta m_1 = \frac{m_0 \gamma}{c} \Delta t \tag{5}$$

$$\Delta m_2 = \frac{m_1 \gamma}{c} \Delta t \tag{6}$$

where m_0 and m_1 denote the instantaneous mass of fuel left in the rocket at a given instant, and are determined by integrating equations (4) and (5).

Composing an unlimited number of equations similar to (4), (5), (6) and so on, integrating them and taking their sum we shall obtain the answer to the proposed problem. For practical purposes it will suffice to take into account the first, second and third supplementary masses of fuel. But in the results of the calculations given above we have also considered the fourth and fifth. It was assumed that $\Gamma' = \gamma' = 10$ m/sec.^{2}, $c' = 1$ km/sec., $S' = 1$ m^2, $C'_x = 0.1$. The density of the air was supposed to conform to the laws of the international standard atmosphere.

Making use of the similarity laws, the results of numerical integration obtained for a particular case can be extended to any case of rocket motion with constant acceleration.

In fact, we may write the following relation for the instantaneous mass of the «main» store of fuel

$$\frac{m_0}{m'_0} = \frac{\Delta m_0}{\Delta m'_0} = \frac{Rc'}{R'c} \frac{\Delta t}{\Delta t'}$$

Now, it follows from (3) that at a given height

$$\frac{R}{R'} = \frac{C_x}{C'_x} \frac{S}{S'} \frac{\Gamma}{\Gamma'} \frac{\sin \beta'}{\sin \beta} \tag{7}$$

since $s = h/\sin \beta$, and

$$\frac{\Delta t}{\Delta t'} = \frac{t}{t'} = \sqrt{\frac{\Gamma'}{\Gamma} \frac{\sin \beta'}{\sin \beta}}. \tag{8}$$

Therefore

$$\frac{m_0}{m'_0} = \frac{C_x S}{C'_x S'} \left(\frac{c}{c'}\right)^{-1} \left(\frac{\Gamma}{\Gamma'}\right)^{1/2} \left(\frac{\sin \beta}{\sin \beta'}\right)^{-3/2} \tag{9}$$

Setting

$$\left(\frac{c}{c'}\right)^{-1} \frac{\gamma}{\gamma'} \left(\frac{\Gamma}{\Gamma'}\right)^{1/2} \left(\frac{\sin \beta}{\sin \beta'}\right)^{-1/2} = K \tag{10}$$

it is easy to find that for the first supplementary consumption of fuel

$$\frac{m_1}{m'_1} = \frac{\Delta m_1}{\Delta m'_1} = \frac{m_0}{m'_0} K \tag{11}$$

and for the n-th consumption

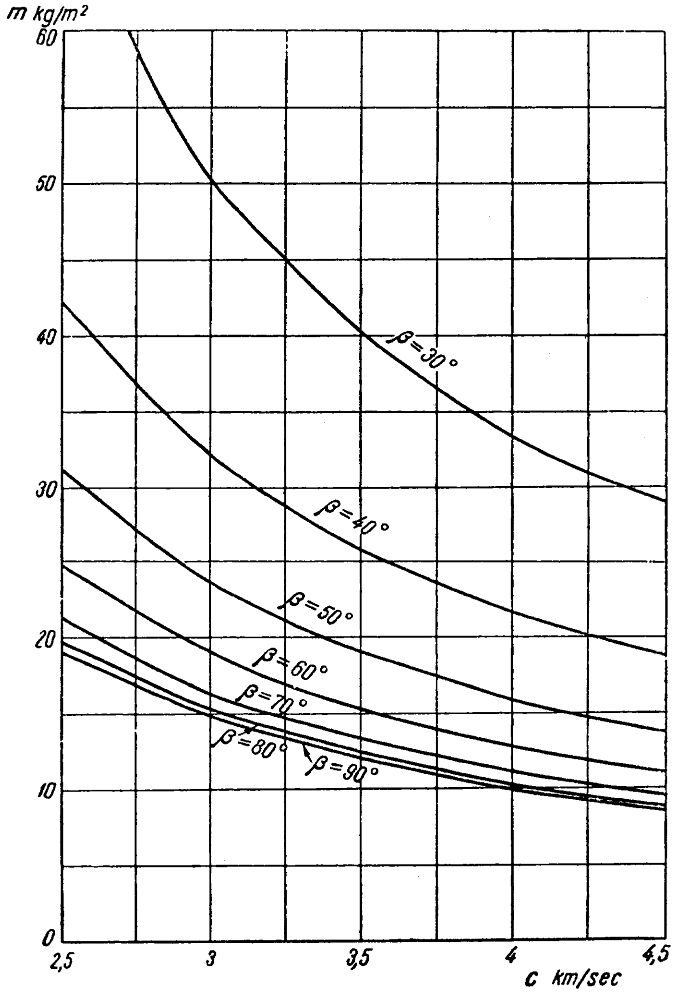
$$\frac{m_n}{m'_n} = \frac{\Delta m_n}{\Delta m'_n} = \frac{m_0}{m'_0} K^n \tag{12}$$

Thus, the similarity law for the total amount of fuel necessary to cross the atmosphere is expressed by the formula

$$m = \frac{C_x S}{C'_x S'} \frac{\gamma'}{\gamma} \frac{\Gamma}{\Gamma'} \frac{\sin \beta'}{\sin \beta} K (m'_0 + m'_1 K + m'_2 K^2 + m'_3 K^3 + \dots) \tag{13}$$

Substituting in this equation the respective values for the vertical start of the rocket ($\beta = 90$) found by numerical integration while using the above-mentioned values for the symbols in equation (13) ([2]), we arrive at the proposed formula (1).

By means of equation (13) and the data given in ([2]) the problem we are interested in can as rapidly be solved for any instant of the flight of the rocket through the atmosphere. This is particularly useful in the case of a step-rocket or a rocket sent from a height h above sea level with a velo-



city $v=\sqrt{2\Gamma h/\sin\beta}$. In the adjoined table are given some values of m_0'/C_x', m_1'/C_x', *etc.*, for $C_x'=0.1$.

h, km	m_0'/C_x'	m_1'/C_x'	m_2'/C_x'	m_3'/C_x'	m_4'/C_x'	m_5'/C_x'
0	182.844	79.580	21.665	4.618	0.843	0.138
2	166.869	43.644	8.164	1.276	0.179	0.023
4	144.341	30.452	4.963	0.706	0.093	0.012
6	120.687	21.818	3.240	0.434	0.055	0.007
8	98.238	15.796	2.197	0.283	0.035	0.004
10	78.087	11.516	1.530	0.192	0.024	0.003
15	41.259	5.442	0.679	0.082	0.009	0.001
20	21.243	2.691	0.327	0.038	0.004	0.000
25	10.764	1.325	0.154	0.017	0.002	0.000
30	5.268	0.598	0.065	0.007	0.001	0.000
35	2.556	0.271	0.028	0.003	0.000	0.000
40	1.230	0.124	0.012	0.001	0.000	0.000
45	0.590	0.057	0.005	0.000	0.000	0.000
50	0.282	0.027	0.002	0.000	0.000	0.000
60	0.065	0.007	0.000	0.000	0.000	0.000
70	0.015	0.002	0.000	0.000	0.000	0.000

Making use of formula (1) and the table of values of Γ/γ((1), p. 182—184), the author has determined the consumption of fuel on the traversing of the atmosphere by rockets with $\gamma=4$ $g_0=39.26$ m/sec^2, starting at an angle β ranging from 30 to 90° (see figure).

It should be noted that in the computations we are concerned with g, and thereby Γ, can be looked upon as constant and equal to their initial values. Verification shows that values obtained on this assumption are overestimated by some hundredths of one per cent only. As for the curvature of the Earth, it can be neglected for not very small angles of start, so that the trajectory of the rocket in the atmosphere may be regarded as a straight line.

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