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An analysis of Newton's projectile diagram

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Abstract. Newton's famous mountain projectile diagram, reproduced in countless science books, is analysed in detail—mathematically, graphically, and historically—perhaps for the first time. A study of the relationship between this diagram from Newton's *On the System of the World* and the written texts on projectile motion in this book and in the *Principia* reveals an ambiguity in Newton's presentation. We propose an explanation of the ambiguity based on an exploration of its roots in Newton's thought.

1. Introduction

It is a well known tale that Isaac Newton's insight into what became his theory of gravity came when he realized that the same (gravitational) force pulling an apple to the Earth may also hold the moon in its orbit. At the time (about 1666) Newton was living on his mother's farm because Cambridge University was closed due to a plague. Historians still have not determined if the apple story is apocryphal or not, but nevertheless it does express Newton's insight.

From a pre-Newtonian viewpoint the idea seems nonsensical; an apple falls vertically to the Earth whereas the moon orbits the Earth in a closed path. (Newton knew the closed path was an ellipse from Kepler's discovery of 1609.) How can these two very different categories of motion be combined? Well, they can if the apple story is altered. What if a stiff wind blows the apple off the tree, so that it falls along an arc? The apple will then be a projectile and, as Galileo discovered and Newton knew, it will appear to fall along a parabolic path, another of the conic sections. Now in the limited case of any real projectiles fired in the 17th century, a parabola on a flat Earth is a sufficient model. But Newton made the conceptual leap to the whole Earth and considered projectiles (which we would call missiles today) shot around the world in elliptical paths (the parabola thus being an approximation to the ellipse). Consequently, the apple and the moon are both falling, the difference being that the moon, so-to-speak, falls forever. Thus Newton was able to join what Aristotelian physics had deemed incongruous categories of motion.

Newton's insight is made manifest in his famous diagram of projectile motion (figure 1) which shows several objects being shot horizontally off a mountain at the north pole of the Earth. This diagram is extensively reproduced in textbooks but seldom, if ever, analysed in any detail. It is sometimes assumed to be taken from the *Principia* (1687) [1]. However, no such 'pictorial' diagrams are to be found in the *Principia*, except for the very last figure in the book (which is Newton's drawing of the path of the great comet of 1680). All diagrams in the *Principia* are abstract geometrical ones. Newton's projectile diagram appears in another book, *On the System of the World* [1, 2], a short popular account of his theory of gravity

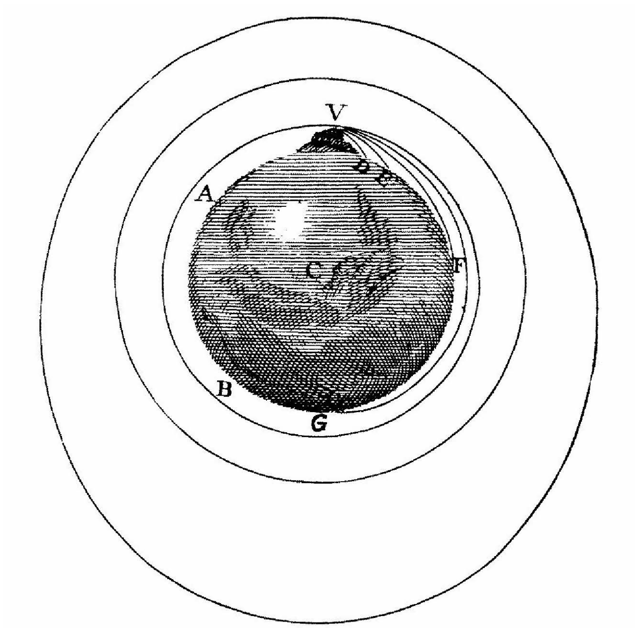

Figure 1. Newton's diagram of projectile motion from his popular *On the System of the World*. From [1, vol II, p 551], reproduced by permission of the University of California Press. This is an engraved representation of the original sketch; see figure 4 for the original.

(written approximately between late 1684 to mid 1685) that originally was intended to be part of the *Principia*. One goal in this paper is to analyse the diagram (figure 1) using a simple descriptive mathematical model. By doing this we will come to appreciate the ambiguity present in Newton's writings on this type of projectile motion, an ambiguity that led several well known scholars astray.

2. The mathematical description

As far as we know, no modern mathematical analysis of the mountain diagram exists. Hence we present a simple description of the projectile motion that is relevant to Newton's diagram. We set up our model such that the projectiles are fired from a raised position corresponding to Newton's mountain (point V in his diagram). The inverse square law of gravity demands that the projectiles follow an elliptical path with the Earth's centre at one of the foci [3]. In terms of standard two-dimensional polar coordinates we have the following equation describing the projectile's motion:

$$r(\theta) = \frac{(R_e + h)(1 - e\cos(\theta_0))}{1 - e\sin(\theta + \theta_0)}. \tag{1}$$

In this equation $r(\theta)$ is the radial displacement of the projectile, θ is the polar angle measured from the horizontal axis, R_e is the radius of the Earth, h is the height of the mountain, e is the eccentricity, and θ_0 represents the angle by which the entire ellipse, being constrained to always pass through the top of the mountain, is rotated if the projectile is not sent off horizontally. Using conservation of angular momentum, the eccentricity can be written in terms of the ratio of the magnitudes of the initial projectile velocity and the velocity necessary to put the projectile into a circular orbit [4]. To model Newton's diagram it is only necessary to consider velocities less than or equal to the velocity required to achieve a fully circular orbit. Newton does not show any other path possibilities. For this class of velocities, v, the eccentricity is as follows:

$$e = 1 - \left(\frac{v}{v_{circle}}\right)^2. \tag{2}$$

The velocity magnitude v_{circle} is given as

$$v_{circle} = \sqrt{\frac{GM_e}{R_e + h}}. \tag{3}$$

Demanding that the projectile strike somewhere on the Earth's surface requires that $r(\theta) = R_e$ for some θ. This constraint gives the following relation:

$$\sin(\theta + \theta_0) = \left(1 + \frac{h}{R_e}\right)\cos(\theta_0) - \frac{h}{eR_e}. \tag{4}$$

The bound on this sine function leads to a bound on the eccentricity for those projectile paths that strike the Earth's surface. By equation (2), this leads to the following restriction on the magnitude of the initial projectile velocity:

$$1 - \frac{(h/R_e)}{(1 + (h/R_e))\cos(\theta_0) - 1} \leqslant \left(\frac{v}{v_{circle}}\right)^2 \leqslant 1 - \frac{(h/R_e)}{(1 + (h/R_e))\cos(\theta_0) + 1}. \tag{5}$$

In Newton's diagram (figure 1), projectiles are fired horizontally off the mountain and therefore $\theta_0 = 0$. With this value of θ_0, the above velocity bound implies that all elliptical projectile paths that get interrupted by the Earth's surface vary between the approximate parabolic curves having velocities close to $v = 0$ and the nearly-circular curves bounded by the maximum velocity

$$v_{max} = v_{circle}\sqrt{\frac{2}{2 + (h/R_e)}}. \tag{6}$$

Substituting this maximum velocity into the eccentricity and calculating θ from equation (4), we find that $\theta = -\frac{\pi}{2}$. Thus Newton's projectiles can hit the surface no further than the south pole (point G in figure 1) when they are fired horizontally from a raised platform at the north pole [5]. To gain perspective on the physical ramifications of the mathematics describing Newton's mountain missiles we will now employ a numerical example. Assume that Newton's north-pole mountain is similar in height to that of Mount Everest and therefore take $h = 8848$ m. By equation (6) we find that $v_{max} = 0.9997\, v_{circle}$ using† $R_e = 6.378 \times 10^3$ m. With $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-1} and $M_e = 5.974 \times 10^{24}$ kg, equation (3) gives the speed necessary to achieve a circular orbit as $v_{circle} = 28\,440$ km h^{-1}. The maximum range, whereby the projectile hits the south pole, is therefore obtained by sending off the projectile at $v_{max} = 28\,430$ km h^{-1}, only 10 km h^{-1} less than that necessary for a circular orbit. Any subsequent augmentation of the horizontal starting velocity, no matter how small, will produce an almost circular elliptical orbit that ends up, unimpeded, back where it started. Since this idea is important for what we discuss later, it is worthwhile to demonstrate it analytically by changing the initial velocity an infinitesimal amount to $v = v_{max} + \delta v$ with δv being the infinitesimal velocity increment. Substituting this new velocity into equation (1) using equation (2) we obtain the following result for the separation, $h_{south\ pole}$, of the projectile from the Earth's surface at the $\theta = -\frac{\pi}{2}$ angle:

$$h_{south\ pole} = \frac{2(\delta v/v_{max})R_e}{(1 - \frac{1}{2}(v_{max}/v_{circle})^2)} \tag{7}$$

† The value for the height of Mount Everest has been taken from [6] and all other constants from [7].

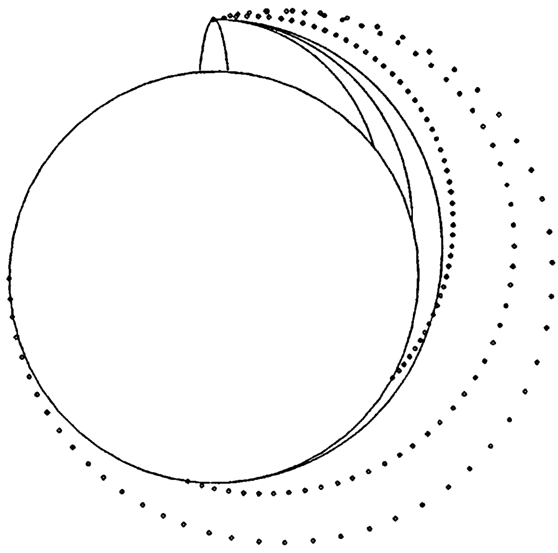

Figure 2. The paths of projectiles leaving a mountain at the north pole of the Earth (assuming no resisting media) are shown. The three solid curves represent the paths of projectiles shot off horizontally with each succeeding path having smaller eccentricity as the initial velocity is increased. The smallest eccentricity possible that still leads to a projectile hitting the Earth is the eccentricity of the almost circular path that just strikes the south pole. The three paths outlined by small squares represent constant eccentricity projectiles that have not been shot off horizontally. The larger the initial angle deviates from the horizontal, the larger the range is for these curves.

Thus the projectile skirts the south pole arbitrarily close to the surface of the Earth when the initial starting velocity is infinitesimally larger than the maximum missile range. Using our numerical Mount Everest example we see that approximately a separation of $h_{south\ pole} = 1$ m would occur if the initial firing velocity is augmented by $\delta v = \frac{1}{1000}$ km h^{-1}. If the augmentation is $\delta v = 1$ km h^{-1}, the separation is approximately one kilometre. Since the eccentricity is still less than one but greater than 0 for these projectiles, the projectile curves are necessarily closed and therefore the fired objects must come back, without interference, to the tip of Newton's mountain. This counterintuitive result is consistent with Newton's diagram since no projectile paths are drawn reaching the Earth beyond G. If the projectile is to land on the side of the Earth opposite from the direction in which it is launched then the projectile must be fired at an angle to the horizontal. This effectively means that the major axis of the projectile ellipse is rotated by some angle $\theta_0 \neq 0$. Projectile paths for this more general case are shown in figure 2 for various values of θ_0.

The above mathematical description, although not in the geometrical style of Newton, is sufficient for an understanding of what Newton could have known about projectile motion.

3. Puzzling text for a clear diagram

In figure 3, a diagram taken from Thomas S Kuhn's classic text *The Copernican Revolution* [8] is displayed. This diagram copies Newton's projectile sketch by showing the transition from parabolic-like paths into more obvious elliptical paths. Path T1 is close to that of a parabola, T2 and T3 are more clearly arcs of ellipses, and T5 is an elliptical orbit. What then is path T4 on Kuhn's diagram? It cannot be an ellipse since, as we have shown in the previous section, no projectiles shot off horizontally can reach the Earth beyond the south pole. In Newton's diagram, figure 1, no path corresponding to the projectile curve T4 in Kuhn's diagram is to be

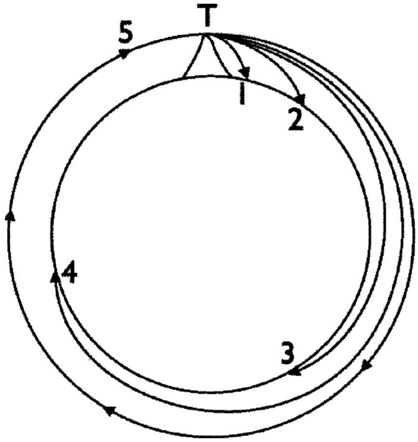

Figure 3. Kuhn's version of Newton's diagram, copied by the authors. Kuhn's diagram does not contain the labels T1–T5, which we added, but we have carefully copied the geometry of the original.

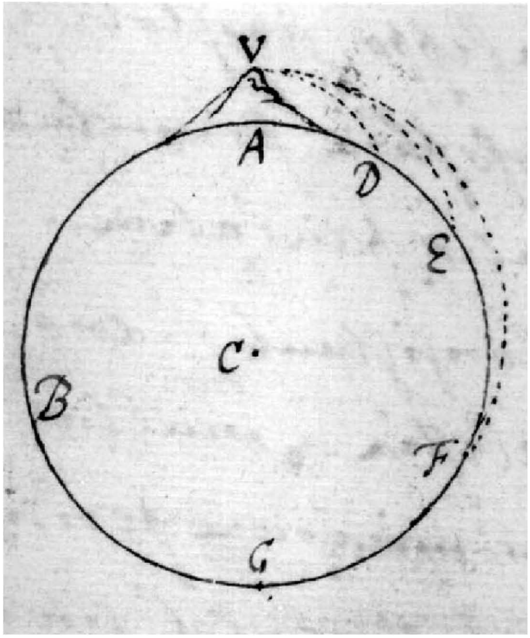

Figure 4. Newton's original sketch of projectile motion made for the popular *On the System of the World* (MS Dd.4.18 f.1v). The original sketch is only about 38 mm in diameter! Reproduced by permission of the Syndics of Cambridge University Library.

found. A non-trivial point about the history of the diagram shown in figure 1 is that Newton did not draw the exact diagram found in *On the System of the World*—that diagram was done by an engraver when the work was published. Fortunately, however, the original sketch by Newton is still extant (figure 4). This sketch clearly shows that Newton did not draw a projectile falling beyond G to the Earth.

The above information seems to indicate that Kuhn's error did not derive from Newton's diagrams and therefore, one would think, not from the ideas as found in the popular *System* or the *Principia*. However, as we will see, maybe Kuhn (and any other scholars who have made a similar mistake [9]), can be partially excused for reading Newton's text too closely. Here, for instance, is what Newton says about the projectile motion in the accompanying text of the *System* [10]:

> Let AFB represent the surface of the Earth, C its center, VD, VE, VF, the curve[d] lines which a body would describe, if projected in an horizontal direction from the top of an high mountain, successively with more and more velocity. And, because the celestial motions are scarcely retarded by the little or no resistance of the spaces in which they are performed; to keep up the parity of cases, let us suppose either that there is no air above the Earth, or at least that it is endowed with little or no power of resisting. And for the same reason that the body projected with a less velocity, describes the lesser arc VD, and with a greater velocity, the greater arc VE, and augmenting the velocity, it goes farther and farther to F and G; if the velocity was still more and more augmented, it would reach at last quite beyond the circumference of the Earth, and return to the mountain from which it was projected.

At first, Newton's argument appears consistent with the diagram: after a projectile reaches point G, any further 'augmented' launch would set it into orbit. A closer reading, however, suggests something else. The phrase 'still more and more augmented' seems suspiciously redundant when an infinitesimal change is sufficient to get a projectile to 'return to the mountain from which it was projected'. Does this redundancy not imply that Newton thinks that points on the Earth beyond G are reached by a projectile? Although, of course, no such 'paths' are to be found in Newton's diagram, and it is a mistake that many people (even established scholars) might make if they do not check out projectile curves mathematically, it is possible that the aforementioned diagram of Kuhn may indeed have been based on his reading of this passage from Newton. Kuhn drew paths to the far side of the Earth assuming that initial velocities 'more and more augmented' would reasonably lead to such paths.

In short, Newton's text in the *System* is cryptic on whether Newton knows that paths to the far side of the Earth for his projectiles cannot exist. And thus there is an apparent ambiguity between Newton's text and the diagram.

What of the *Principia* itself? As mentioned above, no projectile diagram appears, but Newton does discuss the path of such a projectile. It is found in the introductory section, under definition V ('Centrifugal Force'). The passage reads as follows [1, vol I, p 3] (note the military metaphor, not found in the popular *System*, which is probably not unrelated to major improvements in cannons at the time):

> If a leaden ball, projected from the top of a mountain by the force of gunpowder, with a given velocity, and in a direction parallel to the horizon, is carried in a curved line to the distance of two miles before it falls to the ground; the same [projectile], if the resistance of the air were taken away, with a double or decuple [tenfold] velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line which it might describe, till at last it should fall at the distance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or lastly, so that it might never fall to the earth, but go forwards into the celestial spaces, and proceed in its motion *in infinitum*.

As in the passage from the popular *System*, Newton's presentation is puzzling, but here even more so. Seemingly correct at the start, he considers projectiles sent '10, 30, 90 degrees' from the launch point; and, of course, he concludes with a projectile going into orbit 'so that it might never fall to the earth'. However, in-between is the case where the projectile 'might go quite round the whole earth before it falls'. Is this not the case of a path starting at the top of the mountain, encircling the Earth, and terminating at its base? The path T4 in figure 3 would behave exactly like this if it were extended; thus the *Principia* may be an alternative source of Kuhn's error. There does not seem to be any other way to interpret Newton's phrase, even though such an interpretation implies a situation that we have shown cannot arise in a vacuum. It is therefore not surprising that a reader might be misled. Hence, our reading of the ambiguous passage in the *System* is consistent with what we find in the *Principia*.

4. Hedging his bets

It is both interesting and significant to note that although Newton made corrections to the *Principia* whenever he had the opportunity, the ambiguity of the projectile passage was never corrected. The 3rd edition, with various changes, was published early in 1726, about 14 months before Newton's death. Why were there no changes? A key to the answer may be found by taking a look into Newton's thought in the years prior to writing the *Principia*. An important exchange of letters with Robert Hooke, initiated by Hooke in November 1679, stimulated Newton to think more deeply on scientific matters, especially on the subject of celestial motion. The exchange of letters lasted into January of 1680 and, ultimately, set Newton on track to creating the *Principia*. Before this correspondence Newton had scarcely thought about such things as celestial physics since the time of his isolation during the plague years. He had instead devoted considerable effort to theological and alchemical studies. At

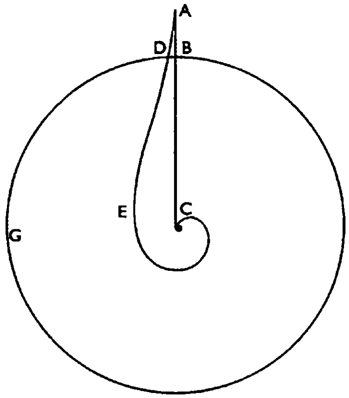

Figure 5. Newton's spiral of November 1679, from his letter to Hooke, copied by the authors. The original drawing by Newton is reproduced in [13]. Note that in this figure the North–South axis of the Earth points out of the page, whereas it is vertical in the other figures.

most, he had speculated about the cause of gravity, employing a mechanism based upon an intuitively sensible all-pervading aether [11]. The aether, as a mechanism for gravity, was the most common astronomical model for what the 17th century called the 'mechanical philosophy'. Any non-mechanical model, such as action-at-a-distance, was considered an unscientific explanation for gravity (or any other force in nature). In the aether model that Newton favoured, borrowed primarily from Descartes, the gravitational field around the Sun arose from a vortex of aether swirling around the Sun.

Hooke's first letter to Newton was on the topic of the paths of falling bodies on the rotating Earth. In his first reply to Hooke (the exchange appears in [12, pp 297–313]), Newton drew a diagram of an object, released from a fixed tower, falling towards the centre of a rotating Earth (figure 5). Newton's diagram indicated that, because the Earth is rotating toward the east, the object would fall to the east (since the initial tangential velocity at the top of the tower is greater than at the bottom); moreover, the diagram further showed that the object would continue to the centre of the Earth following a spiral path. In light of Newton's aether model, Newton's spiral sketch has a conceptual basis since it entailed a vortical aether path.

Hooke replied to Newton with an ingenious argument, based on Keplerian orbits and a split Earth, which clearly showed that a body falling with respect to the Earth's centre would be constrained to fall along an elliptical path. Newton seemed to accept that Hooke's Keplerian point of view had validity for Earth-based projectiles since he did not defend his spiral path in the last letter he wrote to Hooke. In this terse reply Newton acknowledged that the path would not be a spiral but argued that, in a rotating frame, it would not be Hooke's exact ellipse curve either.

In the following years Newton seems to have hedged his bets on which explanation—a gravitational force or an aethereal vortex—he was willing to claim as truth. Thus, in a letter exchange with Thomas Burnet, regarding Burnet's book on a hypothesis for the origin of the Earth, Newton uses the phrase, '...ye pressure of ye vortex or of ye Moon upon ye Waters...' (December 1680); in the next letter (January 1681) he says '...ye pressure of ye Moon or Vortex may promote ye irregularity of ye causes of hills....' In both cases he seems to imply that either the 'vortex' (i.e. an aetherial medium) or the 'moon' (i.e. a gravitational force) are equal in their explanatory power. As well, the next month, in a letter to the Astronomer-Royal John Flamsteed, on the comet of 1680, Newton begins a sentence with, 'The attraction of ye earth by its gravity...,' but a sentence later refers to '...ye motion of ye Vortex...' [12, pp 319–41].

Despite these prior vacillations, however, by the time of the publication of the *Principia*, Newton's mechanistic vortex theory for gravity appears to have been eliminated from his worldview as a consequence of the success of his own analytic studies into the motion of bodies and gravitation. Two examples of this loss of confidence in the vortical theory are as follows. At the end of Book II of the *Principia* (the section on motion in a resisting medium) he proves that

Kepler's third law will not hold if a vortex around the sun is postulated, and he concludes that 'the hypothesis of vortices is utterly irreconcilable with astronomical phenomena. . . .' [1, vol I, p 396] Also, in a letter to Halley in July 1686 he speaks of Hooke 'correcting my Spiral. . . ' [12, p 447]. This admission is surprising since it was certainly not in Newton's demeanour to acknowledge his own mistakes and he was especially not predisposed to give credit to any corrections coming from Hooke. Indeed, over 30 years after the letter exchange with Hooke, Newton tried to dismiss the spiral he drew as 'a negligent stroke' of his pen—which it most surely was not [14].

So why do the projectile passages in the *System* and the *Principia* imply a projectile path that analytically cannot exist? Our analysis suggests that Newton never did make a rigorous calculation of the bounds of projectiles striking the Earth. Instead, assuming that the extrapolation to large ranges was transparent, he relied on his intuition, a physical intuition that had preferentially chosen a vortex theory for gravity earlier in his life. While the results of analytic work superseded this vortical theory whenever Newton applied himself to a problem, it may very well have subtly crept back into his thinking when he felt a calculation might not be needed. Hence he may not have realized, even to the day of his death, that a projectile fired horizontally from the north pole can never land on the Earth past the south pole.

Acknowledgments

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References

[1] Newton I 1687 *Principia: Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his System of the World* vols I and II, transl. A Motte, rev. F Cajori 1966 (Berkeley, CA: University of California Press)
[2] Cohen I B 1971 *Introduction to Newton's 'Principia'* (Cambridge: Cambridge University Press) pp 109–15, 306–8, 327–35
[3] Scarborough J B 1945 The actual path of a projectile in a vacuum *Am. J. Phys.* 13 253–5
[4] Fowles G R 1986 *Analytical Mechanics* 4th edn (New York: Saunders College) pp 148–9
[5] Baker C D and Hart J 1955 Maximum range of a projectile in a vacuum *Am. J. Phys.* 23 253–5
[6] Stott C and Twist C 1995 *Space Facts* (Willowdale, Canada: Firefly) p 83
[7] Cohen E R 1995 *The Physics Quick Reference Guide* (Woodbury, NY: American Institute of Physics) pp 53–4
[8] Kuhn T S 1957 *The Copernican Revolution: Planetary Astronomy in the Development of Western Thought* reprinted 1985 (Cambridge, MA: Harvard University Press) figure 51 on p 255
[9] Koestler A 1964 *The Sleepwalkers: A History of Man's Changing Vision of the Universe* (Harmondsworth: Penguin) p 513
Toulmin S and Goodfield J 1961 *The Fabric of the Heavens: The Development of Astronomy and Dynamics* (New York: Harper Torchbooks) p 234
[10] Newton I 1685 *A Treatise on the System of the World* 2nd English translation 1731, reprinted 1969 (London: Dawsons of Pall Mall) p 6
[11] Dobbs B J T and Jacob M C 1995 *Newton and the Culture of Newtonianism* (Atlantic Highlands, NJ: Humanities Press) pp 20–56
Whiteside D T 1970 Before the *Principia*: the maturing of Newton's thought on dynamical astronomy, 1664–1684 *J. Hist. Astron.* 1 5–19
[12] Turnbull H W (ed) 1960 *The Correspondence of Isaac Newton* vol II (1676–1687) (Cambridge: Cambridge University Press)
[13] Lohne J A 1967 The increasing corruption of Newton's diagrams *History of Science* 6 74, figure 4
[14] Westfall R S 1980 *Never at Rest: A Biography of Isaac Newton* (Cambridge: Cambridge University Press) p 385