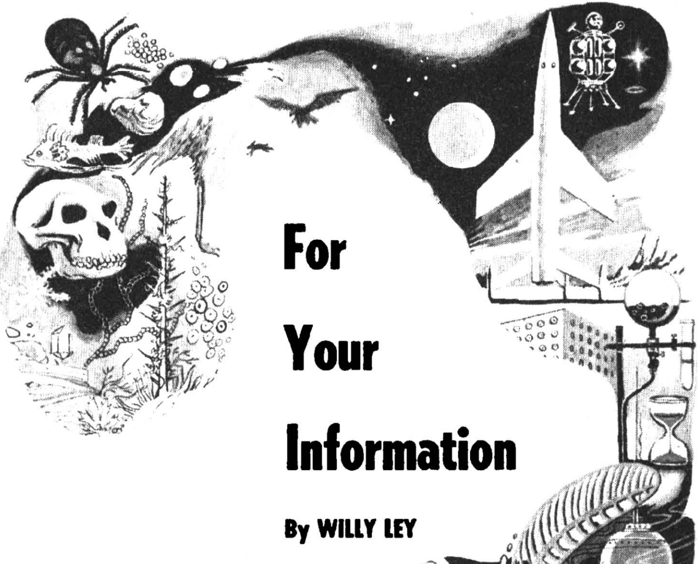
Galaxy SCIENCE FICTION

JUNE 1953

35¢

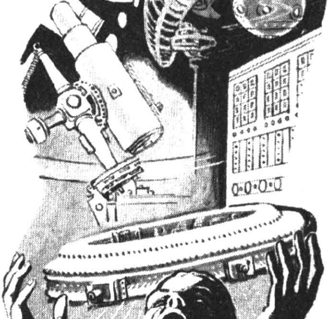
ANC

TANGLE HOLD By F. L. WALLACE



For Your Information

By WILLY LEY



THE INTRACTABLE PRIME NUMBER

JUST about two centuries ago there lived in my father's town of Koenigsberg a gentleman by the name of Christian Goldbach. He was a mathematician by profession and a good one, though not an outstanding genius or a great innovator in his science. When he died in 1764 he left a theorem behind which is in itself simple enough to be understood by a school child. It does not even need to be expressed as a formula to be stated concisely.

It simply says that every even number can be written as the sum of two prime numbers.

Try it. Any two prime numbers from "3" on up will always result in an even number when added together. Nor will you be able to find an even number that cannot be expressed as the sum of two primes.

If the figure you pick is not too small, it can probably be expressed as the sum of two primes in a number of different ways. For example, 100 can be written as 3 + 97, or as 11 + 89, 17 + 83, 41 + 59, 47 + 53, and a number of other ways.

The trouble with Goldbach's beautiful and simple theorem is that it cannot be proved. Trying it out on any number of figures is no proof, if only because the amount of even numbers is infinite. A proof would consist of a sentence, or a paragraph, or, if necessary, a book which shows why this *has* to be so. Goldbach himself could not furnish such proof and later mathematicians, both professional and amateur, who went after the problem had to give up sooner or later in total defeat.

WHEN attacking such a riddle, it is often useful to establish all the characteristics of the units involved. If we knew enough about the mathematical laws governing prime numbers, we might find the one which is responsible for the workability of Goldbach's theorem. That alone might be the reason why we can't prove it, for if there are specific laws governing the occurrence and structure of prime numbers, we haven't found them yet.

But let's go over the ground systematically and begin with the definition of a prime number. That's easy—a prime number is a whole number which does not have any divisors. Or to put it into the stricter language of number theory: "An integer p which is larger than 1 is called a prime number if the only divisors are the trivial ones $\pm$ 1 and $\pm$ p."

In this phraseology, the "1" is not considered a prime number, but many number theorists count "1" as a prime number, too. They are all agreed that "2" is a prime number, the only even prime number in existence. It is also the only prime which—for just this reason—must be left out when using Goldbach's theorem. The next number "3" is also a prime and so are "5," "7," and "11." After them follow "13," "17" and "19."

A method for establishing prime numbers has come down to us from antiquity and is known as the Sieve of Eratosthenes. To use it, you write down all the numbers in the interval you wish to investigate, say from "1" to "100." Then you cross out all even numbers except the "2," for all even numbers are divisible by "2." Next you cross out all remaining numbers which are divisible by "3," except the "3" itself. They are easy to recognize even when very large, because all you have to do is add their digits together.

If your number is 13,623, you form the sum $1 + 3 + 6 + 2 + 3 = 15$, with "15" divisible by "3," which means that the figure itself is divisible by "3." Then you cross out all numbers ending with a "5" for they are divisible by "5" (the numbers ending in a "0" which are also divisible by "5" are already gone since they are also even) and then you proceed to those divisible by "7." It can easily be seen that you use successively larger primes, as you establish them, to knock out all the multiples of these primes. The process is tedious, but reliable.

The next question is how many prime numbers there are. By means of the Sieve of Eratosthenes, it has been established that there are 26 prime numbers in the first hundred; 168 in the first thousand; 303 in the first two thousand; 78,498 in the first million and 50,847,478 in the first thousand million. These figures show what one would suspect, anyway; namely, that the primes get rarer as we move up the ladder into larger and larger figures.

If we go sufficiently high, will we come to a point where there are no primes left?

THE answer is no and again the proof has come to us from antiquity. Euclid himself established it. Supposing somebody claimed that the figure *Z* were the largest and last of the prime numbers, we then take all the smaller primes and multiply them, beginning with 1 times 2 times 3 times 5 times 7 times 11 and we go through all of them until we have reached the prime below Z. Then we multiply the whole by Z, obtaining a much larger figure which we call P. Because of the way we constructed this figure, we can be certain that P is *not* a prime. But how about $P + 1$?

Well, $P + 1$ either is a prime or it isn't. If it is, we have a prime much bigger than Z. If $P + 1$ is not a prime, it can only be divisible by an unknown prime which must be smaller than P, *but larger than Z*. So whether $P + 1$ is a prime or not, it proves that Z cannot be the largest prime.

No matter how high we climb in the realm of numbers; every once in a while we are bound to encounter a prime.

This, of course, brings up the question about the largest prime actually known. It is a really monstrous figure, even though it can be written down in a simple form: $2^{127} - 1$. This, at any event, is easier to memorize than the same figure in arithmetic long hand where it reads:

170,141,183,460,469,231,731,687,-303,715,884,105,727.

How $2^{127} - 1$ was found is a small story in itself. At this point, it is important that this figure had to be found and then established as a prime. It could not be constructed, for there is no method of constructing large primes. Quite a number of mathematicians have tried to find such a method, partly because they might have been bored with the tedious Sieve of Eratosthenes; mostly, however, because a method of constructing primes would be a version of one of the mathematical laws governing prime numbers.

Probably the first attempt to find such a method was that of Pierre de Fermat, a Frenchman who lived from 1601 to 1665 and who, although technically an "amateur," was one of the great mathematicians of history.

Pierre de Fermat believed that the expression

$$2^{(2n)} + 1$$

where "n" successively assumes the value of 1, 2, 3, 4, etc., could be used for the construction of primes. Not of all primes, of course, but he thought that every figure constructed in accordance with that expression would be a prime. In action, it would look like this:

$n=0$,	therefore	$2^{1}+1=3$
$n=1$,	"	$2^{2}+1=5$
$n=2$,	"	$2^{4}+1=17$
$n=3$,	"	$2^{8}+1=257$
$n=4$,	"	$2^{16}+1=65,537$
$n=5$,	"	$2^{32}+1=4,294,967,297$

and up to 65,537 these figures actually are primes.

The figure for $n=5$ was also believed to be a prime for at least a hundred years. But then the great Leonhard Euler, who lived at about the same time as Goldbach, found this "prime" could be obtained by multiplying 6,-700,417 by 641.

FERMAT'S failure to find the limitation of his own method makes one wonder whether he and his compatriot and contemporary Marin Mersenne actually possessed a (lost) method for recognizing large primes instantly. The existence of such a method had been suspected because of the following:

One day, Fermat received a letter asking whether 100,895,-

598,169 was a prime. Reportedly, Fermat replied without hesitation that the figure was not a prime, but the product of two primes, 898,423 and 112,303. Nobody knows how he could tell so fast, but the assumption of a method for recognizing large primes is not convincing in the light of the breakdown of his own formula.

At this point I have to devote a few lines to Monsieur Marin Mersenne, who engaged in correspondence with all the mathematicians of his time and was practically a one-man mathematical clearing house. If he were alive now, he would undoubtedly be the editor of a mathematical journal. In one of his books there is a short paragraph about numbers of a certain kind which have come to be called "Mersenne's Numbers," even though most mathematicians suspect that they are really Fermat's. But since Fermat's name is associated with several other things, the designation with Mersenne's name is quite useful. The numbers in question are numbers of the form $2^p - 1$ and if this expression reminds you of the largest prime known, you are correct; it is one of "Mersenne's numbers."

Boiled down to a minimum of words, Mersenne's statement reads that numbers of the form $2^p - 1$ are primes if p is 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 or 257. In writing this down, Mersenne (or his printer) made an error — the figure 67 should read 61. Also, the two numbers 89 and 107 should have been included. Checking this statement was, of course, a terrible grind, especially in the higher numbers. The low ones are easy: M_1 (this is the way Mersenne's numbers are usually designated) is simply "1," M_2 is "3," M_3 is "7," M_5 is "31." As the rule says neither M_4 nor M_6 is a prime, the former is "15" and the latter "63," both divisible by "3."

M_7 is larger than a hundred, namely 127; M_{13} is 8,191, M_{17} is 131,071 and M_{19} is 524,287. All these numbers were established as early as 1600 by an Italian named Cataldi. The method he used was to divide the number to be checked by all prime numbers smaller than the square root of the number under investigation.

Leonhard Euler established that M_{31} is a prime — it is, of course, easy to find their numerical values, but hard to show what these numbers are once you have them. Two mathematicians, Pervouchine and Seelhoff, established the nature of M_{61} and incidentally that it should be M_{61} and not M_{67}, and the American mathematician R. E. Powers added M_{89} to the list (in 1911) and three years later M_{107}. The next higher Mersenne prime is M_{127} or $2^{127} - 1$, the largest prime known and verified. The verification was done quite some time ago, by the French mathematician Lucas in 1876, but it was not written out numerically until the 1920's.

THOSE who have a lot of spare time and wish to amuse themselves by constructing a few large primes might work with two expressions which, within limits, can be used for this purpose. One reads $n^2 - n + 41$. Here n, as usual, is 1, 2, 3, 4, and so forth. It works for all values of n up to 40. When n grows to be 41, the expression defeats itself, for it then becomes $41^2 - 41 + 41$ which is obviously 41^2 and, being a square, cannot be a prime. Similarly, the expression $n^2 - 79n + 1601$ produces primes only if n equals 79 or less; it collapses when n reaches 80.

Let's see now how far we have progressed. All we really know is that there is no limit to the number of primes. Everywhere else we have failed — we don't have a method of constructing primes and we don't have a simple method of recognizing one when we find it. We can't even tell how many primes there must be between, say, 1111 and 8888.

A certain formula says that the number of primes between 1 and X resembles the figure you get when you divide X by its own "natural" (or basis e) logarithm. But the result only resembles the truth. For X = 100, the formula yields 21.7 instead of 26; for X = 1000, it produces 145 instead of 168, while for X = 1,000,000, the figure is 72,382 and should be 78,498. So the best you can expect is a general guess — all the hard work has to be done just as if that formula did not exist.

Now let's look at something else. Low in the realm of numbers you have quite an assortment of "pairs of primes." Typical examples are 11:13, 17:19, 29:31, 41:43 and 101:103, two primes separated by one (even) number only. We know that primes must occur, if rarely, no matter how large the figures. How about prime pairs? Nobody can tell; there is no proof either way.

Nor is there any rule for the interval between primes. Low down, between "1" and "23," the intervals are small. The first interval over five numbers occurs between "89" and "97," the first interval of more than ten numbers between "113" and "127." Naturally the intervals grow larger as the figures grow; the largest known interval occurs between 4,652,353 and 4,652,507, an interval of 154 numbers. It is almost needless to add that there is no known rhyme or reason to those intervals, either.

In short, in spite of centuries of mathematical effort, the prime numbers are still as intractable as they were in the time of Euclid.

As for Goldbach's theorem, I was still taught, some twenty-five years ago, that nobody had succeeded in doing anything about it. That has changed to some extent. According to George Gamow, two Russian mathematicians have made some inroads on the problem. One of them, Schnirelman, proved in 1931 that each even number is the sum of not more than 300,000 primes. Now this might be an interesting mathematical development, but the outsider has the distinct impression that "not more than 300,000 primes" isn't much help.

A dozen years or so after Schnirelman came Vinogradoff, proving that every even number can be expressed as the sum of four primes. But Goldbach's assertion that it can be done with two primes has been found correct for every case actually tried.

There are mathematical honors to be had for the man or woman who can do better than Vinogradoff and prove *why*.

ANTIKHTHON

PRESUMABLY prompted by a comic strip currently appearing in a number of newspapers, several letters ask whether it might not be possible that there is a planet circling the Sun in the same orbit as the Earth, in such a manner that the Sun is always between us and that other planet so that we'll never be able to discover it. These letters were answered as they came in. This item is intended to forestall those that haven't been written yet.

To begin with, the idea of a planet "at the other end" of Earth's orbit is not precisely new. It was invented more than two thousand years ago by the Pythagorean philosophers who saw an urgent mystical need for another body in the Solar System.

It has to be remarked first that they had something they called the "central fire" of which the Sun was said to be merely a reflection. Now if you counted the bodies in the "Universe" (Solar System to us), you had the "central fire," Sun, Moon, Mercury, Venus, Mars, Jupiter and Saturn —eight altogether. Adding the Earth made nine. But the figure nine was not acceptable to the Pythagoreans. There had to be ten bodies and the way out was to postulate the existence of a "counter-Earth" or *Antikhthon*. Needless to say, no other philosophical school paid the faintest attention to this innovation.

But somehow the idea seems to be intriguing.

For an answer, one could point

out that there is no "counter-Mars" and no "counter-Venus" or "counter-Jupiter." We could see those if they existed. But this, it must be admitted, is no logical proof. Saturn's rings are real even though no other planet in the Solar System has rings. So we are forced to approach this problem from a different angle.

If there were a planet like the ancient *Antikhthon,* we could never see it. This does not mean, however, that we could not detect its presence. Forever hidden in the glare of the Sun though it would be, the effects of its gravitational field would not stay hidden. It should influence the orbit of Venus.

Venus, if there were such an unknown planet, would "unaccountably" deviate from the orbit calculated for it. In fact, the behavior of Venus would enable us to calculate the mass of the unknown planet, especially since we would know its distance from the Sun, which would have to be 93 million miles, like our own.

But Venus fails to show any such "perturbations." As a result, we can take it for granted that the *Antikhthon* does not exist.

—WILLY LEY

ANY QUESTIONS?

Why does a full Moon look larger on the horizon than it does at its zenith? It seems the exact opposite should be true, as the Moon on the horizon is farther away than the Moon overhead.

Charles Tisinger
Bks. S-32, N.A.T.T.C.
Memphis 15, *Tennessee*

This question is a perennial —and let's admit it right in the first sentence—unsolved problem. It is not an astronomical problem, for, as reader Tisinger points out, the Moon overhead is closer to the observer. The difference is on the order of 4,000 miles which, compared to the Moon's distance of 240,000 miles, is so little that it would not be apparent to naked-eye observation.

The problem is somewhere in the realm of either physiology or psychology and probably a mixture of both.

Some have tried to explain it by saying that the phenomenon is caused by the position of the head. If you watch the Moon on the horizon, you look straight ahead, whereas, if you watch it near the zenith, you have to lean back. If that were the whole answer, the phenomenon should disappear by making both observations lying down. One man actually claims that it did disappear under these circumstances. Maybe it worked for him, but it does not work for me.

Another explanation says that you have nothing with which to compare the size of the Moon when it is high in the sky, while you do have objects (trees, buildings, hills) for comparison near the horizon. In my opinion, this should tend to make it look smaller at the horizon, but I know the Moon looks bigger.

In short: explanation unsatisfactory.

I have read that so-called shooting stars are new comets passing through our solar system. If this isn't true, what are they?

Larry Willey
55 Dawson Avenue
Mansfield, Ohio

Shooting stars are tiny particles entering our atmosphere from space, at speeds ranging anywhere from 7 to 45 miles per second. They are so tiny that you could not feel them if you held some in the palm of your hand. Compressing the atmosphere in their paths, they are heated to such an extent that the vast majority of them vaporize. Only a small fraction of their number reach the ground at all.

Far from being "new comets," they are most likely debris of very old comets which slowly disintegrated in their orbits many thousands of years ago.

How did Halley know the time of his comet's return on a 75-year schedule? Did he live to see his comet appear twice? And how is it that comets have never collided with the planets?

Shaler Morris
Veteran's Home, Calif.

Dr. Edmond Halley saw "his" comet only once, in 1682, when he was about 26 years old. He had just gone over a list of comet observations and, while there had been many comets at unpredictable intervals, his suspicion was aroused by the fact that a large and bright comet had been seen at 75-year intervals. There had been one in 1531, one in 1607 and his own observation in 1682. He felt confident that this was the same comet, moving in a closed orbit, and predicted its return for the year 1758. Incidentally, the comet was not labeled "Halley's Comet" until the prediction was verified by actual observation.

As for the third question, the answer is that we do not know whether a comet has ever collided with a planet. We know it has not happened during the last few centuries, but we cannot be positive about the past.